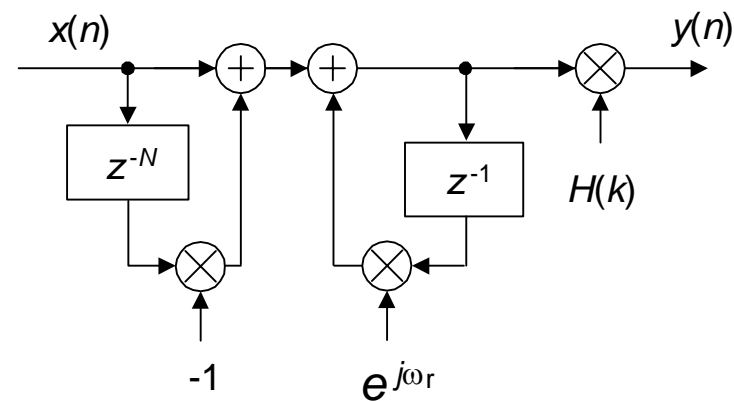


2004 COMP.DSP Conference; Cannon Falls, MN, July 29-30, 2004

Frequency Sampling Filters: The Lost Art

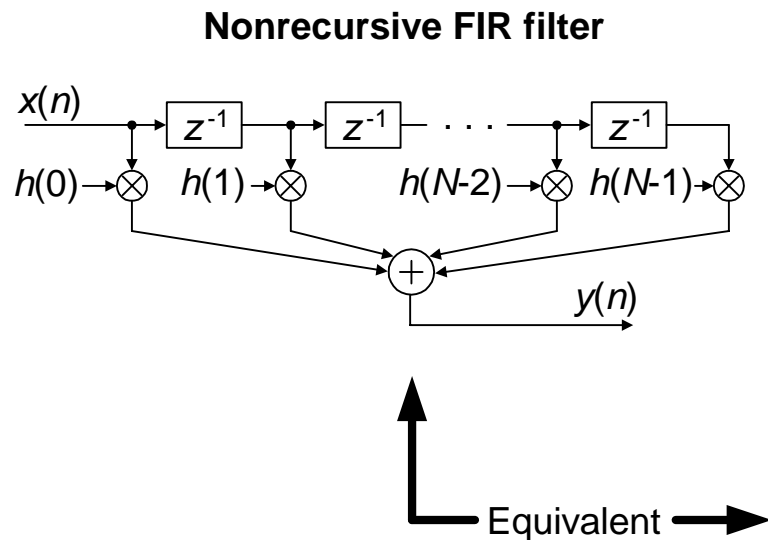
Speaker: Richard Lyons
Besser Associates
E-mail: r.lyons@ieee.com



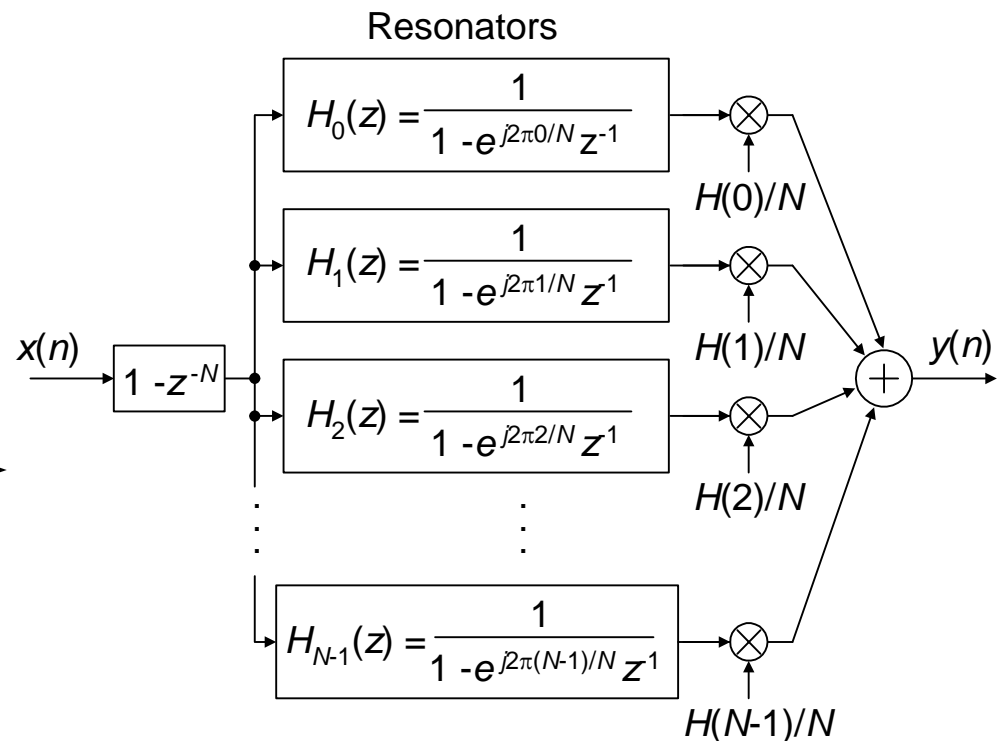
Frequency Sampling Filters

- ▶ *Frequency sampling filters* are used to implement linear-phase FIR filters.
- ▶ Can be more efficient than Parks-McClellan-designed filters for certain applications,
 - when passband $\leq f_s/4$. (f_s is the sampling rate.)
- ▶ Rarely covered in DSP textbooks or classrooms.
- ▶ Here we'll spread the gospel of frequency sampling filters (FSFs).

- ▶ Frequency sampling filters were founded upon the equivalency of:
 - a traditional N -tap nonrecursive (direct convolution) FIR filter, and
 - a comb filter in cascade with a bank of N complex resonators.
- ▶ Coefficients $h(k)$ and $H(k)$ gain factors assumed to be complex.
- ▶ $H(k)$ gain factors are the discrete Fourier transform of the $h(k)$.



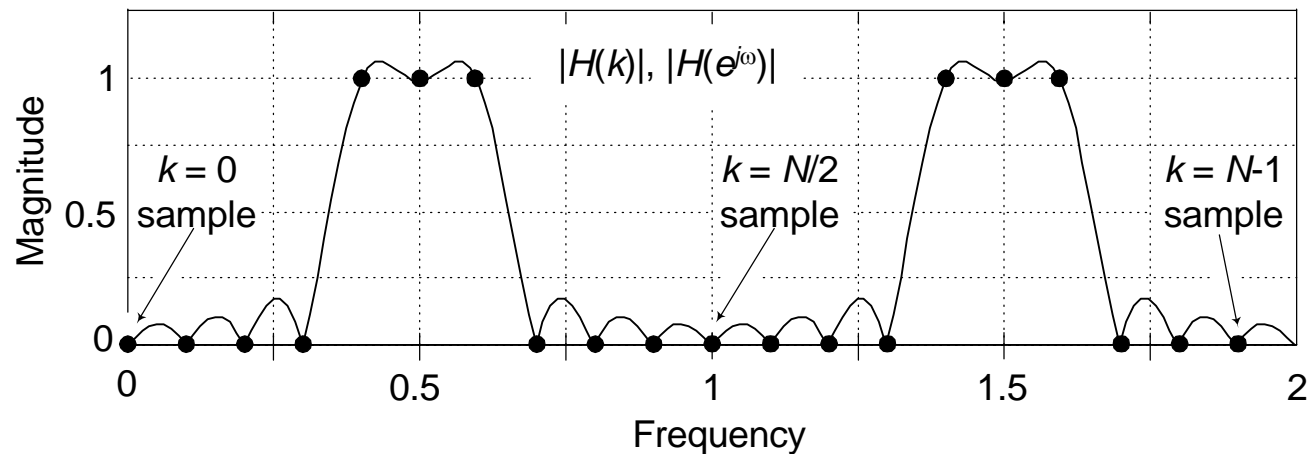
**Frequency sampling filter
(Recursive FIR filter)**



- ▶ **Basis of FSF design is the definition of a desired FIR filter frequency response**
 - in the form of $H(k)$ frequency-domain samples,
 - whose magnitudes are depicted as dots in the figure below.

- ▶ $H(k)$ sample values are used as gain factors
 - following the resonators in the FSF structure.

- ▶ Frequency axis measured in π radians,
 - 2π radians f_s (sample rate in Hz).

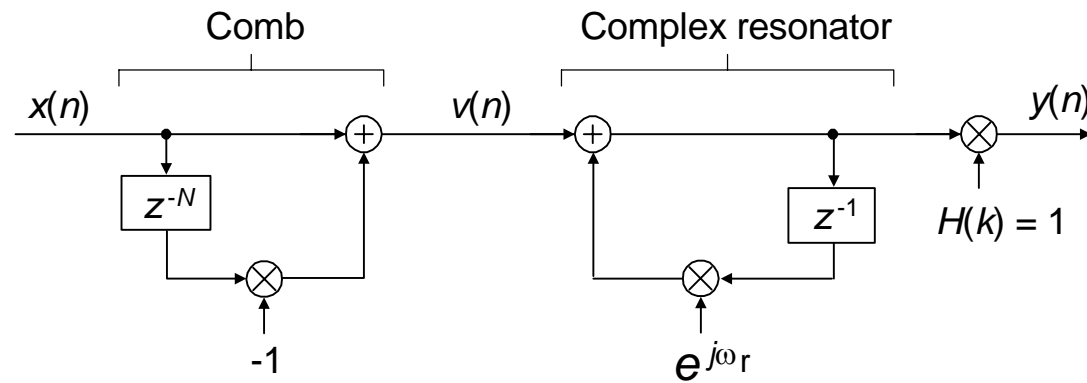


► FSFs are efficient because most $H(k)$ are zero-valued,

- corresponding to the stopband, and
- need not be implemented.

► Simplest FSF is a single section of a complex FSF:

- a comb filter followed by a single complex digital resonator.



► The $1/N$ gain factor following a resonator in a standard FSF is omitted, for simplicity.

- The $1/N$ scaling factor will be discussed later.

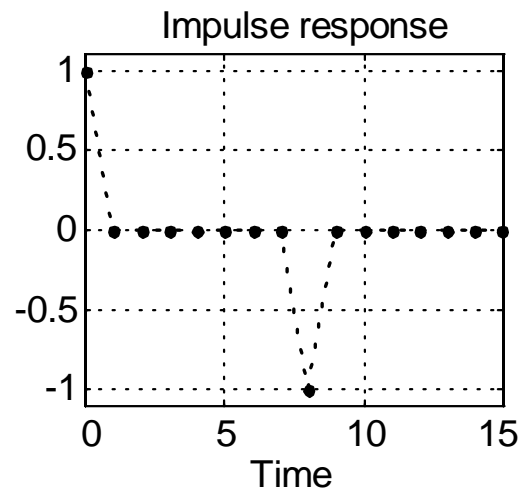
► First, consider nonrecursive comb filter whose time-domain difference equation is

$$v(n) = x(n) - x(n-N)$$

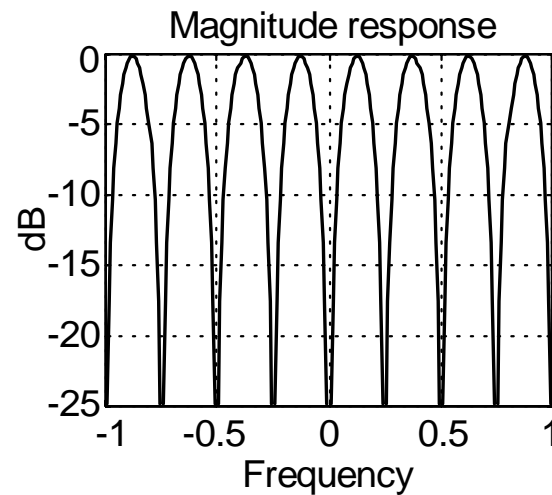
- z-domain transfer function and frequency response are:

$$H_{comb}(z) = \frac{V(z)}{X(z)} = 1 - z^{-N}, \quad \text{and} \quad H_{comb}(e^{j\omega}) = e^{-j(\omega N - \pi)/2} 2\sin(\omega N/2).$$

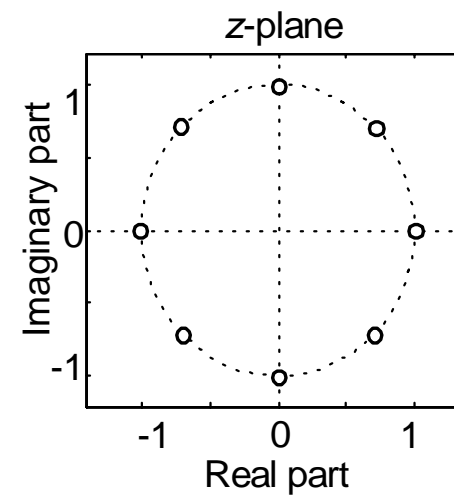
► For $N = 8$, behavior is as shown below.



(a)

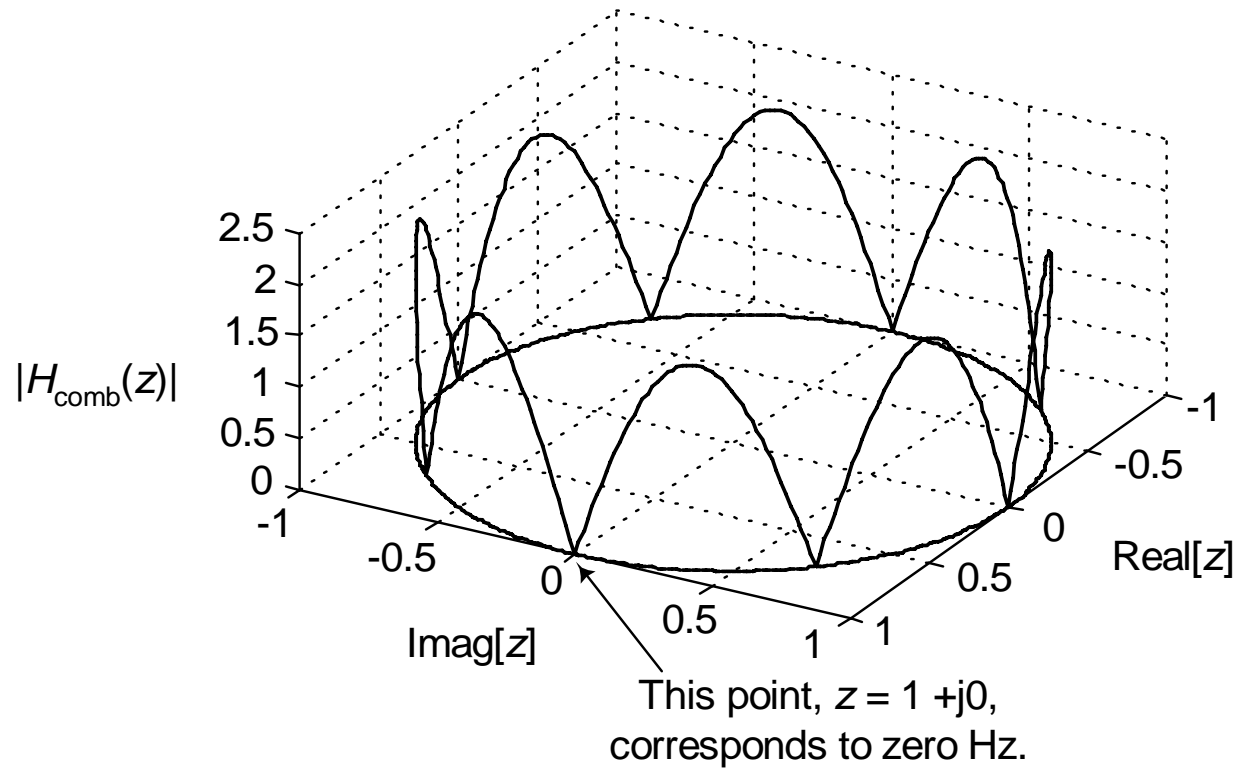


(b)



(c)

- ▶ $H_{\text{comb}}(z)$ has N periodically-spaced zeros around z -plane's unit circle,
 - located at $z(k) = e^{j2\pi k/N}$, where $k = 0, 1, 2, \dots, N-1$.



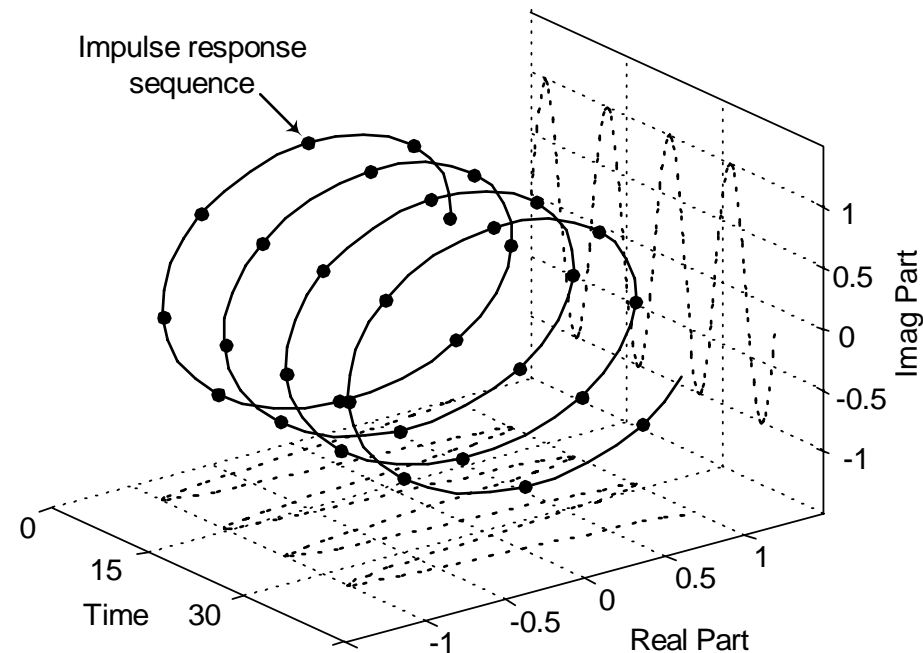
► Now, for the complex resonator:

- Its time-domain difference equation and transfer function are:

$$y(n] = v(n] + e^{j\omega_r} y(n-1), \text{ and } H_{\text{res}}(z) = \frac{Y(z)}{V(z)} = \frac{1}{1 - e^{j\omega_r} z^{-1}}$$

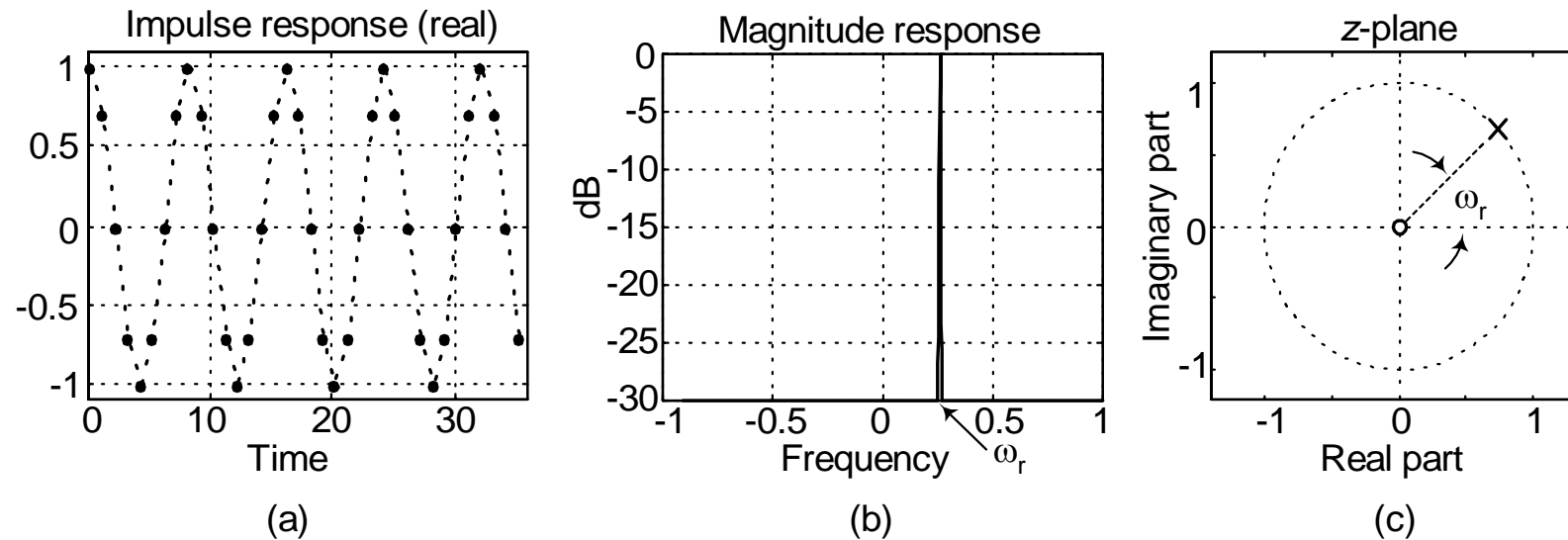
- where the angle ω_r , $-\pi \leq \omega_r \leq \pi$, determines the resonant frequency.

► Resonator's impulse response, for $\omega_r = \pi/4$, is:



► Impulse response is a complex sinusoid.

► $H_{\text{res}}(z)$ has a pole at $z = e^{j\omega_r}$, on the unit circle at an angle of $\omega_r = \pi/4$.



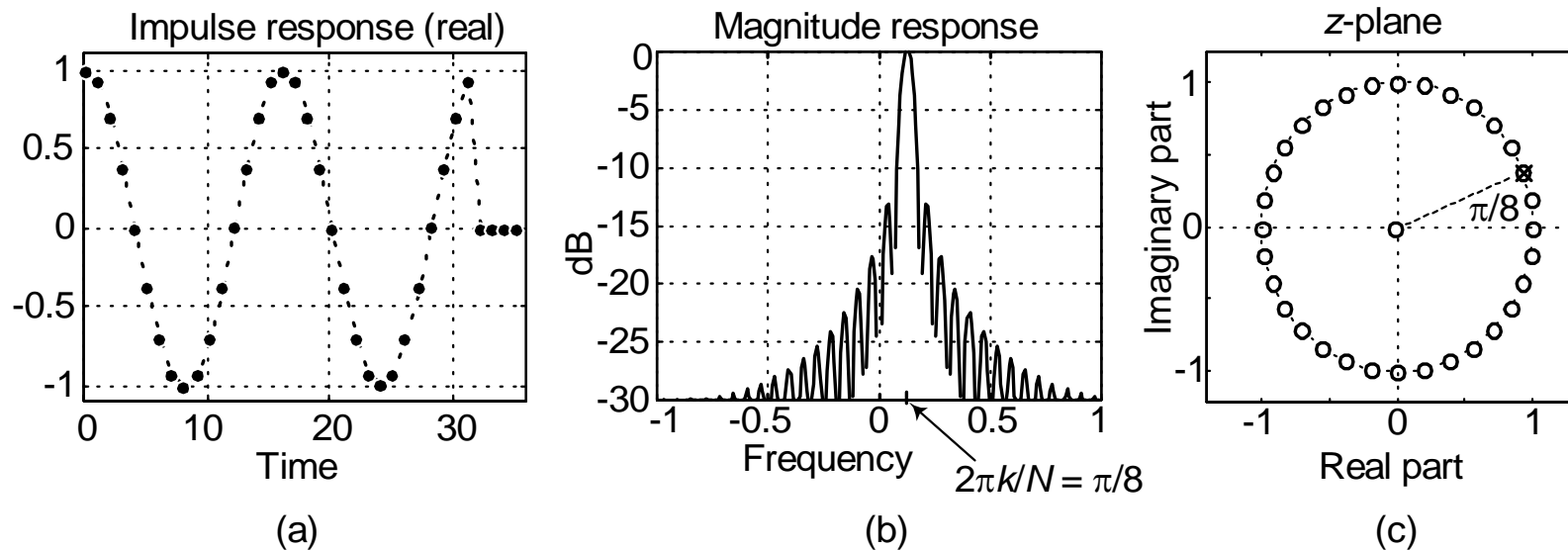
Transfer function of single-section complex FSF is

$$H(z) = H_{\text{comb}}(z) H_{\text{res}}(z) H(k) = (1 - z^{-N}) \frac{H(k)}{1 - e^{j\omega_r} z^{-1}}.$$

- ▶ **Restrict resonator's resonant frequency ω_r to be $2\pi k/N$,**
 - resonator's pole located atop one of the comb's zeros, yielding
 - FSF transfer function of

$$H_{ss}(z) = (1 - z^{-N}) \frac{H(k)}{1 - e^{j2\pi k/N} z^{-1}}$$

- ▶ **For $N = 32, k = 2$, and $H(2) = 1$, FSF behavior is:**

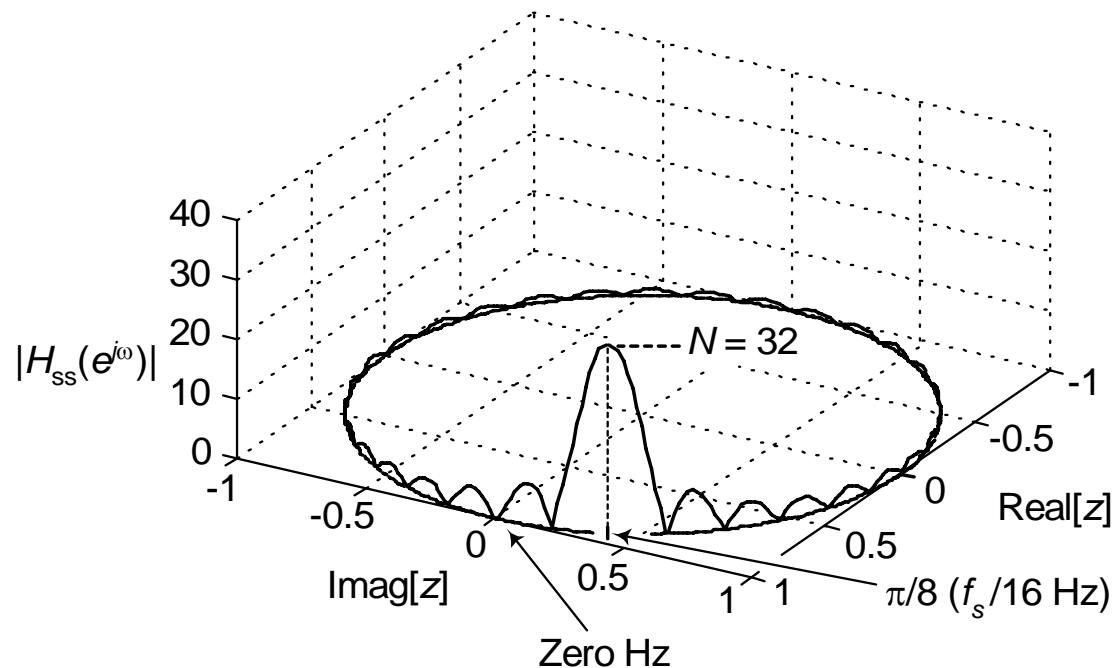


- ▶ **Impulse response is a truncated complex sinusoid.**

- ▶ Frequency magnitude response is a $\sin(x)/x$ -like function.
- ▶ Pole/zero cancelation at $2\pi k/N = \pi/8$ radians ($f_s/16$ Hz).
- ▶ Frequency response is:

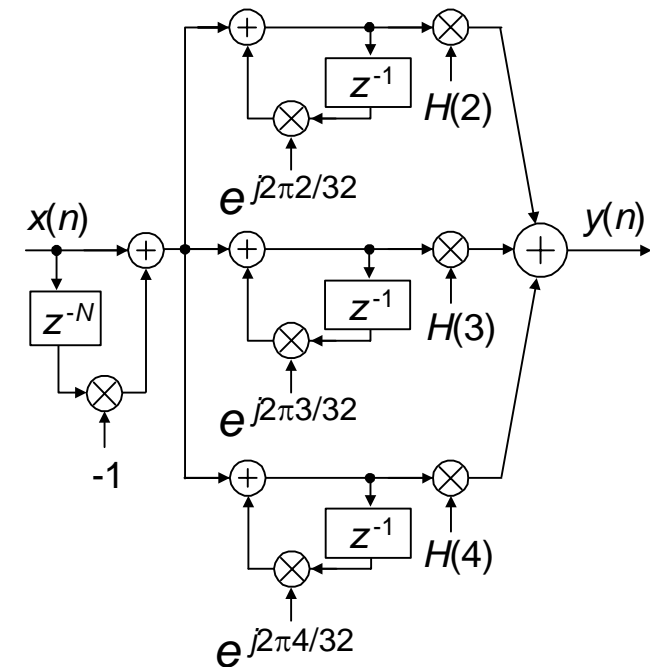
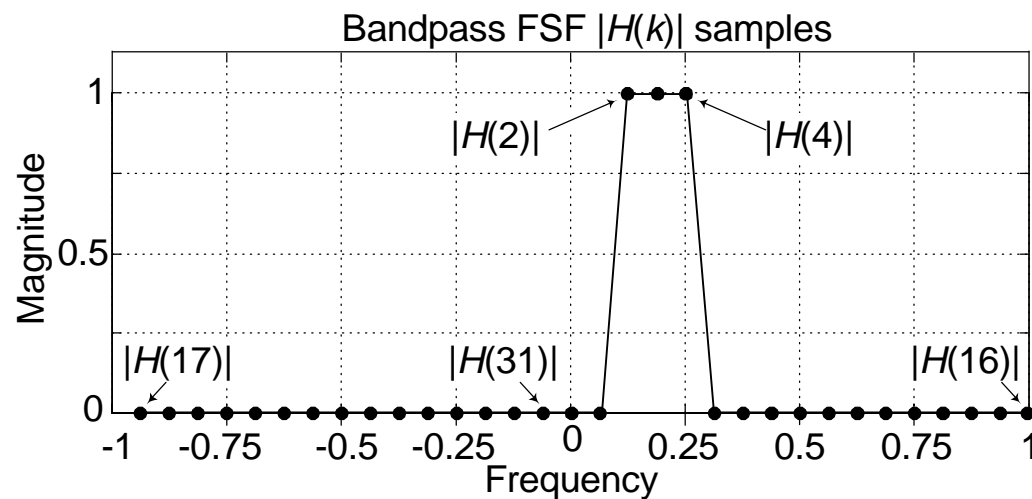
$$H_{ss}(e^{j\omega}) = H_{ss}(z)|_{z=e^{j\omega}} = e^{-j\omega(N-1)/2} e^{-j\pi k/N} H(k) \frac{\sin(\omega N/2)}{\sin(\omega/2 - \pi k/N)}.$$

- ▶ $H_{ss}(e^{j\omega})$ has linear phase.
- ▶ Max magnitude response of $H_{ss}(e^{j\omega})$ is N when $|H(k)| = 1$.



Multisection Complex FSFs

- Practical FSFs use multiple resonator sections,
 - like the following bandpass filter.
 - Here's a 3-section complex FSF: $N = 32$, $H(2) = H(3) = H(4) = 1$.



- Transfer function of the 3-section, $N = 32$, complex FSF is

$$H_{\text{cplx},3}(z) = (1 - z^{-32}) \left[\frac{H(2)}{1 - e^{j2\pi 2/32} z^{-1}} + \frac{H(3)}{1 - e^{j2\pi 3/32} z^{-1}} + \frac{H(4)}{1 - e^{j2\pi 4/32} z^{-1}} \right] = (1 - z^{-32}) \sum_{k=2}^4 \frac{H(k)}{1 - e^{j2\pi k/32} z^{-1}}$$

- ▶ Transfer function of a *general* complex FSF is

$$H_{\text{cplx}}(z) = (1 - z^{-N}) \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{j2\pi k/N} z^{-1}}.$$

- ▶ For an $N = 32$ complex FSF we could have up to 32 resonators,
 - but in practice only a few resonators are needed for narrowband filters.

- ▶ Frequency response of a multisection complex FSF is

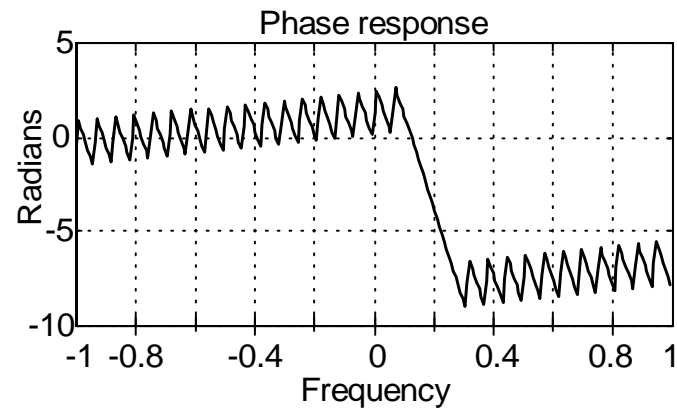
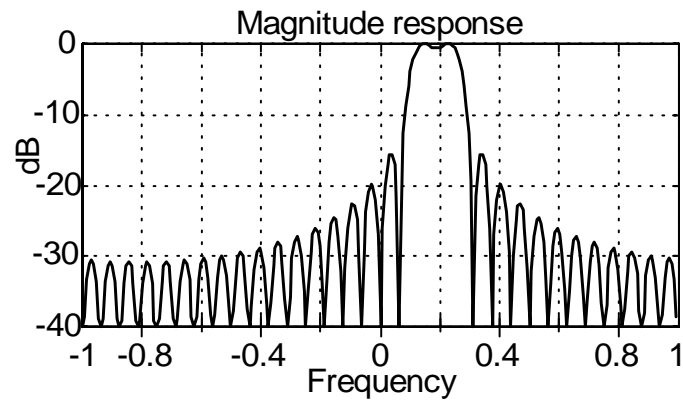
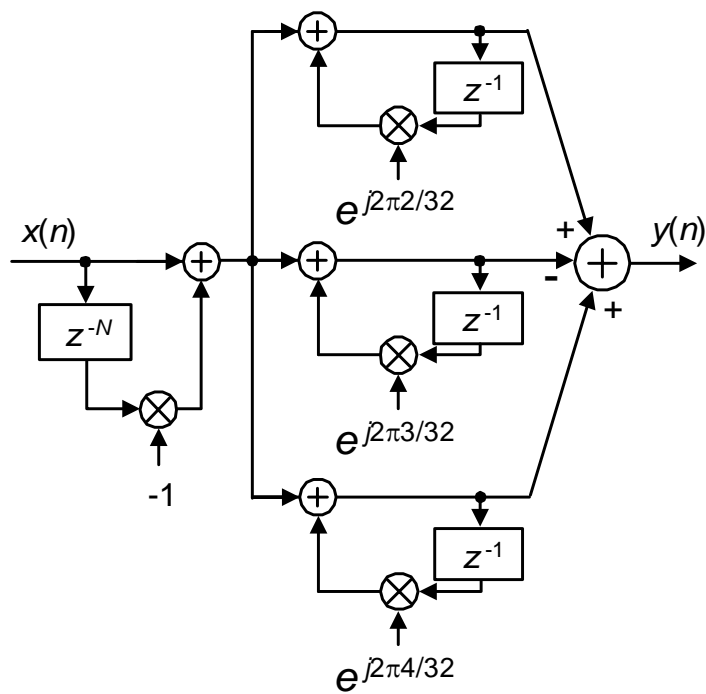
$$H_{\text{cplx}}(e^{j\omega}) = e^{-j\omega(N-1)/2} \sum_{k=0}^{N-1} \frac{H(k) e^{-j\pi k/N} \sin(\omega N/2)}{\sin(\omega/2 - \pi k/N)}.$$

- ▶ Here's a fascinating part of FSFs: to achieve phase linearity,
 - we use $|H(k)|$'s equal to alternating + & - ones, $(-1)^k$.
 - Yielding a transfer function of

$$H_{\text{cplx,lp}}(z) = (1 - z^{-N}) \sum_{k=0}^{N-1} \frac{(-1)^k}{1 - e^{j2\pi k/N} z^{-1}}. \quad (\text{'lp' subscript means "linear-phase"})$$

► The $(-1)^k$ factors simplify our FSF implementation.

- Complex $H(k)$ multiplies are replaced by simple adds and subtracts.



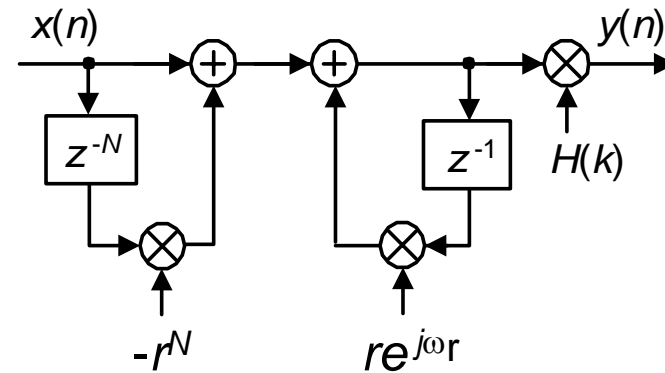
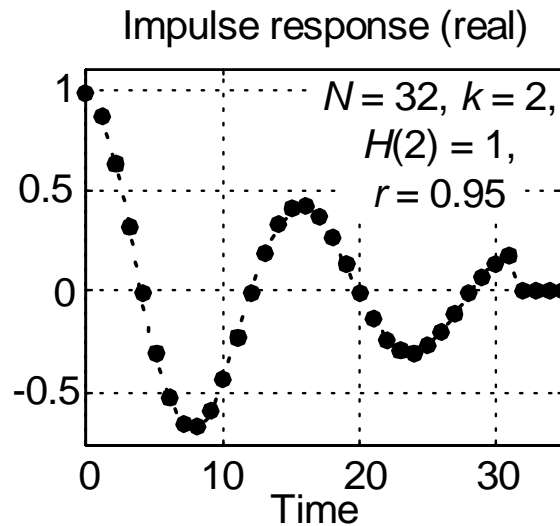
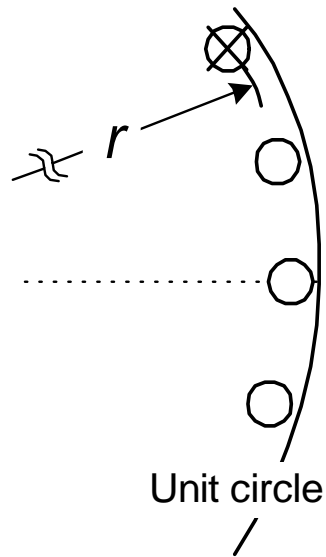
- ▶ **A linear-phase, even- N , complex FSF frequency response is:**
 - **the sum of individual resonators' $\sin(x)/x$ -like frequency responses, and**
 - **expressed with the wicker-looking equation:**

$$H_{\text{cplx,lp}}(e^{j\omega}) = e^{-j\omega(N-1)/2} \sin(\omega N/2) \left[\frac{|H(N/2)|e^{-j\pi/2}}{\sin(\omega/2 - \pi/2)} + \sum_{k=0}^{(N/2)-1} \frac{(-1)^k}{\sin(\omega/2 - \pi k/N)} - \sum_{k=(N/2)+1}^{N-1} \frac{(-1)^k}{\sin(\omega/2 - \pi k/N)} \right].$$

- ▶ **This expression is not as complicated as it looks.**
 - **First term inside the brackets represents the resonator centered at $k = N/2$ ($f_s/2$).**
 - **The first summation are positive-frequency resonators.**
 - **Second summation are the negative-frequency resonators.**
- ▶ **Reference [1] provides the derivation of the $H_{\text{cplx,lp}}(e^{j\omega})$ expression.**

Ensuring FSF Stability

- ▶ **Poles and zeros are like real estate: Location, location, location!**
- ▶ We can ensure filter stability by moving filter's zeros and poles just inside the unit circle,
 - at a radius of r , where the damping factor r is just slightly less than 1.
- ▶ For example, when $N = 32$, $k = 2$, $H(2) = 2$, and $r = 0.95$, we have



- ▶ The transfer function of a guaranteed-stable single-section complex FSF is

$$H_{\text{gs,ss}}(z) = H_{\text{comb},r<1}(z) H_{\text{res},r<1}(z) H(k) = (1 - r^N z^{-N}) \frac{H(k)}{1 - [re^{j2\pi k/N}]z^{-1}} \cdot$$

- Subscript 'gs,ss' means a guaranteed-stable single-section FSF.

- ▶ The z-domain transfer function of a guaranteed-stable N-section complex FSF is

$$H_{\text{gs,cplx}}(z) = (1 - r^N z^{-N}) \sum_{k=0}^{N-1} \frac{H(k)}{1 - [re^{j2\pi k/N}]z^{-1}}$$

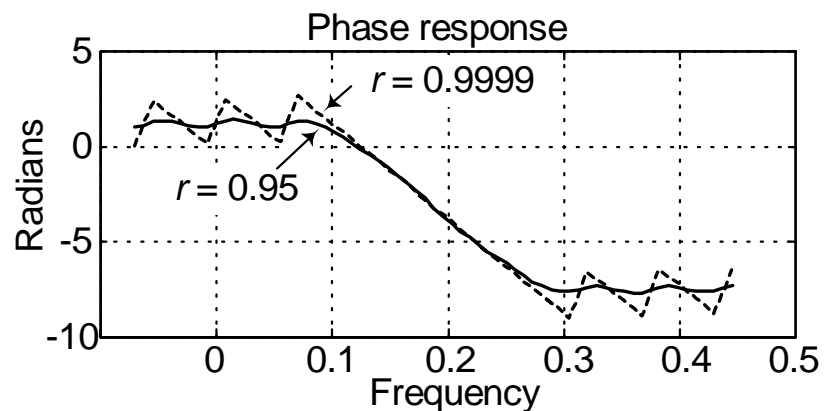
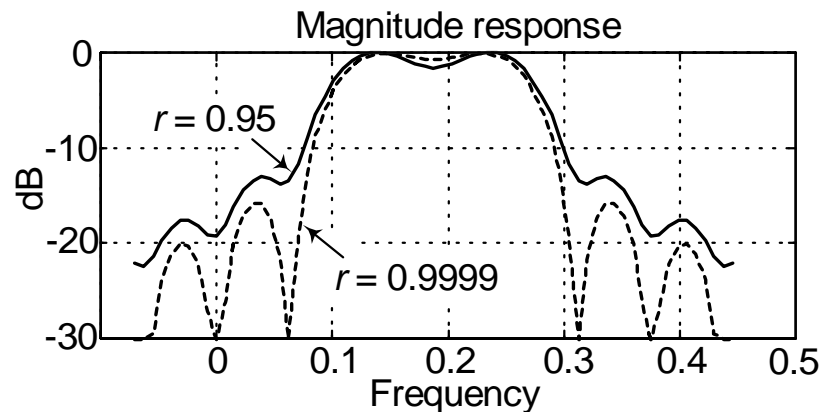
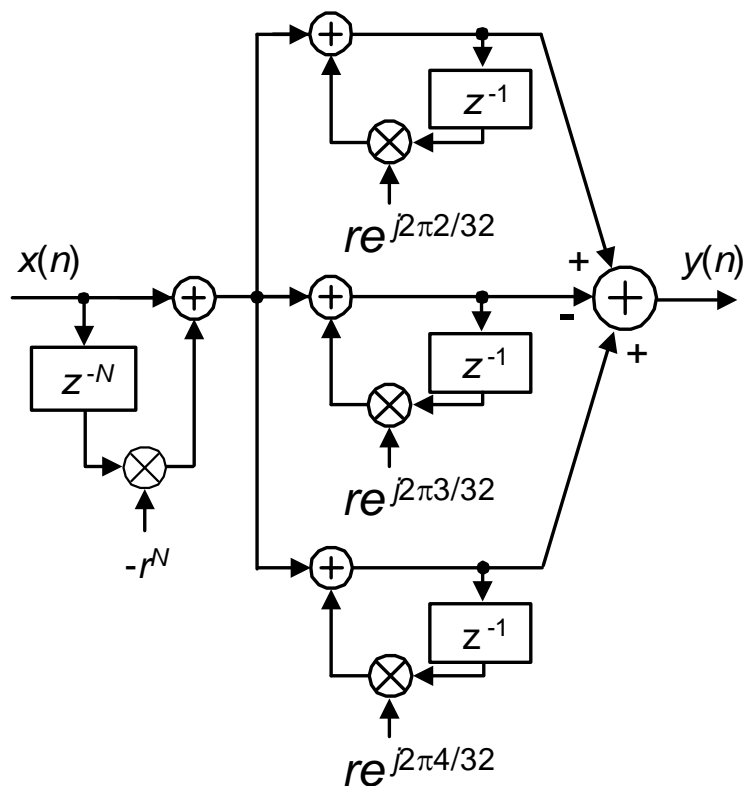
- Subscript 'gs,cplx' means a guaranteed-stable complex multisection FSF.

- ▶ The frequency response of $H_{\text{gs,cplx}}(z)$ is the *messy*

$$H_{\text{gs,cplx}}(e^{j\omega}) = \sqrt{r^{N-1}} e^{-j\omega(N-1)/2} \sum_{k=0}^{N-1} \frac{H(k) e^{-j\pi k/N} \sinh[N \ln(r)/2 - jN\omega/2]}{\sinh[\ln(r)/2 - j(\omega - 2\pi k/N)/2]} \cdot$$

► Effects of zeros and poles inside the unit circle are shown below

- for the two cases where $r = 0.95$ and $r = 0.9999$.



► Define r to be as close to unity as your binary number format allows. If integer arithmetic is used, set $r = 1 - 1/2^B$ where B is the number of bits used to represent a filter coefficient magnitude.

Multisection Real-Valued FSFs

- ▶ Obtain real-FSF structures (real-valued coefficients) by forcing conjugate poles,
 - by ensuring all non-zero $H(k)$ gain factors satisfy $H(N-k) = H^*(k)$.

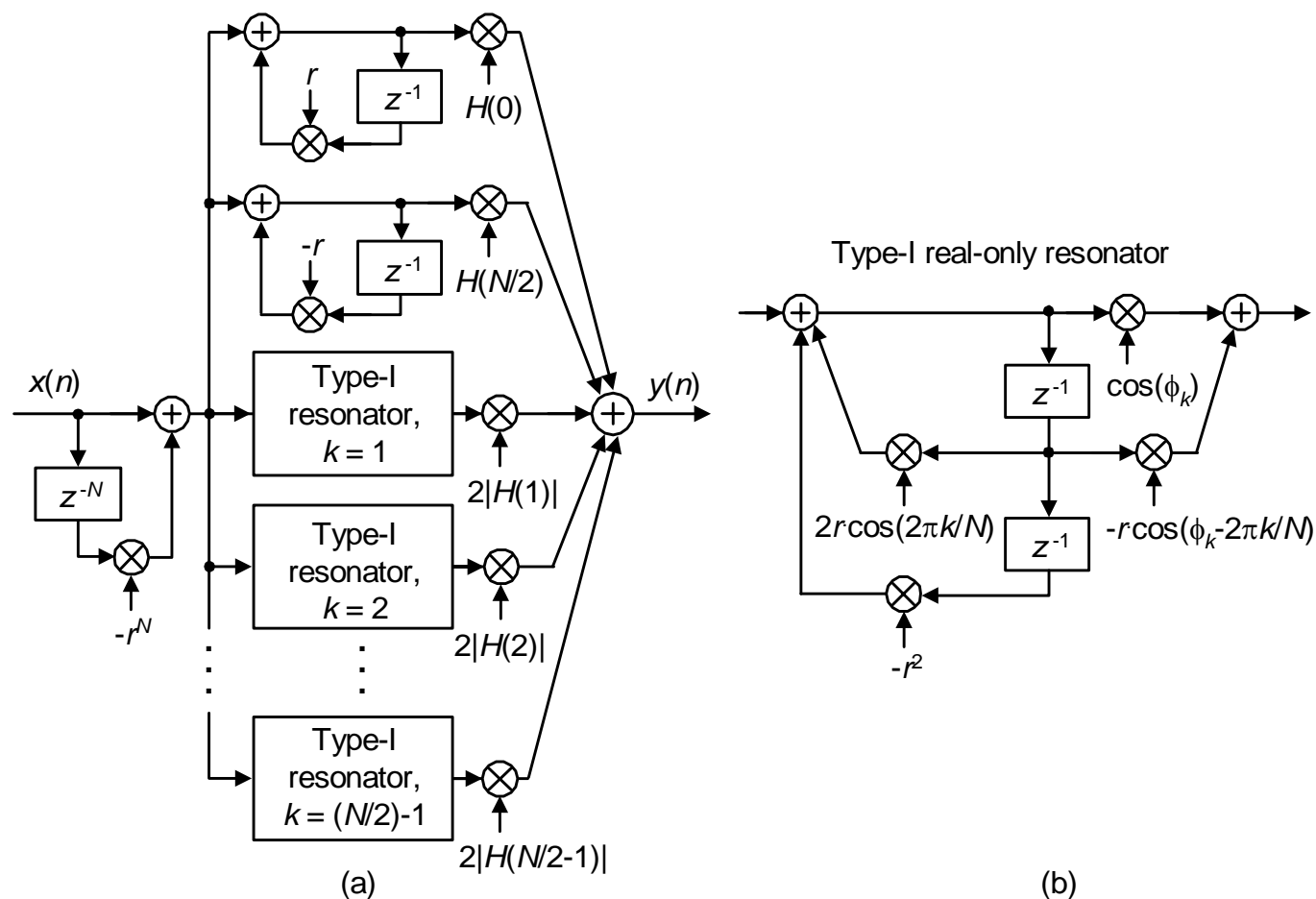
- ▶ Conjugate pole pairs located at angles of $\pm 2\pi k/N$ radians.

- ▶ Transfer function of an even- N real-valued FSF is

$$H_{\text{gs,real}}(z) = H_{\text{Type-I}}(z) = (1-r^N z^{-N}) \left[\frac{H(0)}{1-rz^{-1}} + \frac{H(N/2)}{1+rz^{-1}} + \sum_{k=1}^{N/2-1} \frac{2|H(k)| [\cos(\phi_k) - r \cos(\phi_k - 2\pi k/N) z^{-1}]}{1 - [2r \cos(2\pi k/N)] z^{-1} + r^2 z^{-2}} \right]$$

- Subscript 'gs,real' means a guaranteed-stable real-valued multisection FSF
- ϕ_k is the desired phase angle of the k th section.

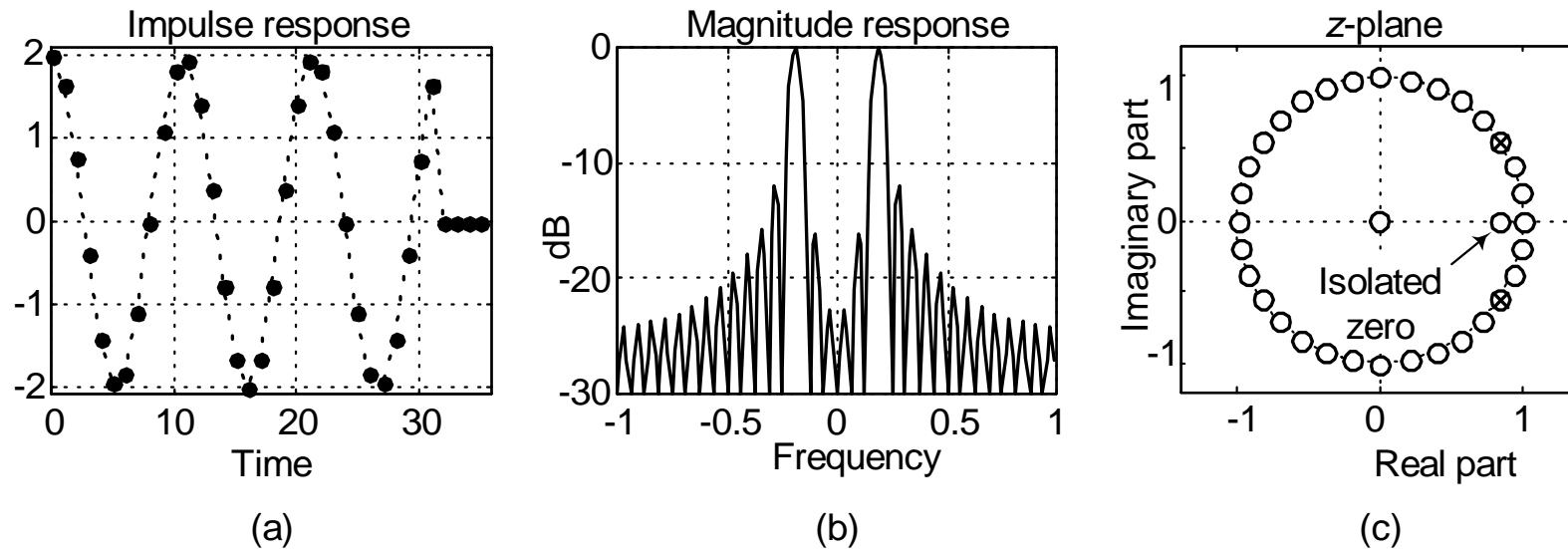
► We call this real-valued FSF a **Type-I real FSF** whose implementation is:



► For lowpass FSFs the stage associated with the $H(N/2)$ gain factor would not be implemented.

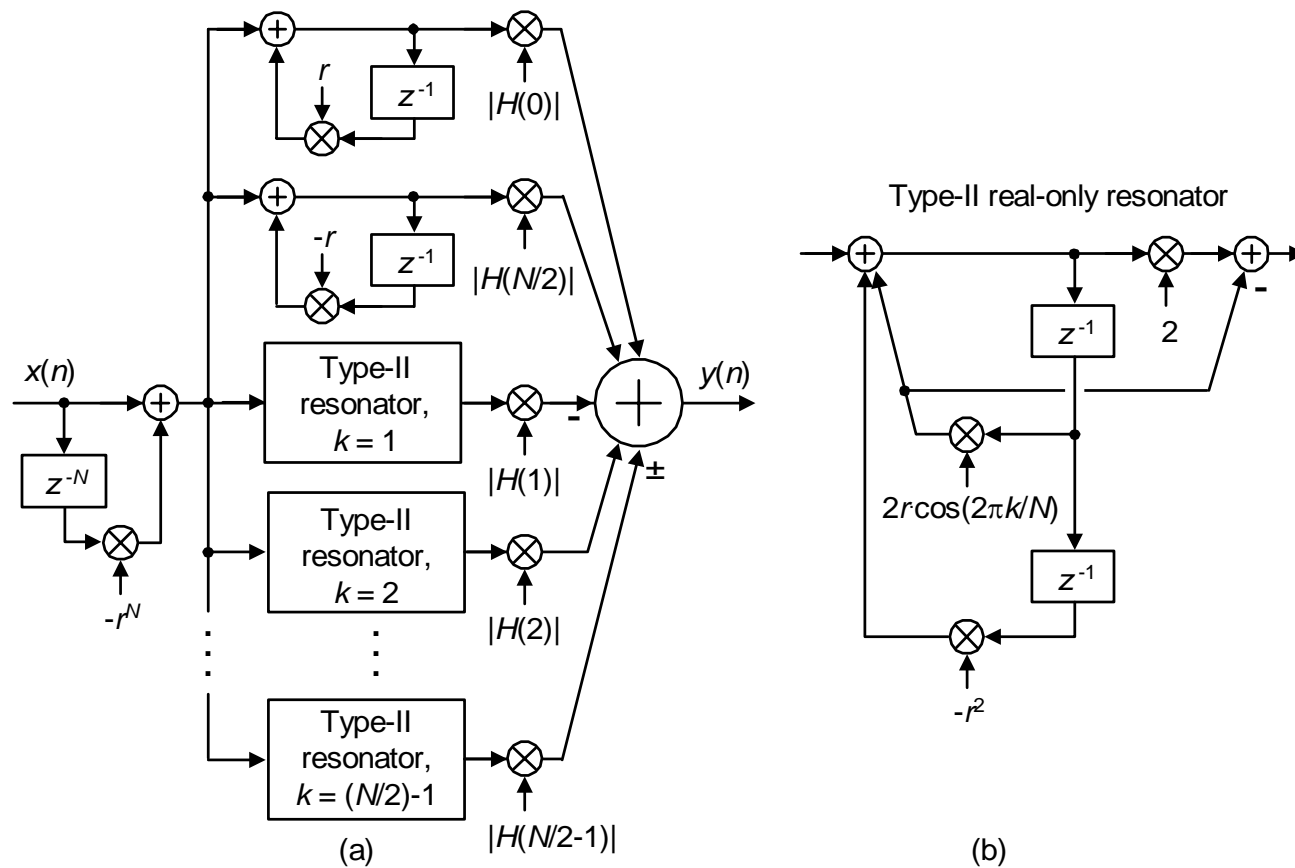
► For bandpass FSFs neither stage associated with the $H(0)$ and $H(N/2)$ gain factors would be implemented.

- The behavior of a single-section Type-I real FSF with $N = 32$, $k = 3$, $H(3) = 1$, $r = 0.99999$, and $\phi_3 = 0$ is :



- An alternate version of the Type-I FSF, with a simplified resonator structure, can be developed:
- Set all ϕ_k values equal to zero,
 - move the gain factor of 2 inside the resonators, and
 - incorporate the alternating signs in the final summation to linearize phase.

► This gives us a **Type-II real FSF** whose implementation is:



► **Type-II resonators require fewer multiplies than do Type-I resonators.**

- ▶ The transfer function of this Type-II real FSF is

$$H_{\text{Type-II}}(z) = (1-r^N z^{-N}) \left[\frac{|H(0)|}{1-rz^{-1}} + \frac{|H(N/2)|}{1+rz^{-1}} + \sum_{k=1}^{N/2-1} \frac{(-1)^k |H(k)| [2-2r\cos(2\pi k/N)z^{-1}]}{1-[2r\cos(2\pi k/N)]z^{-1}+r^2 z^{-2}} \right].$$

- ▶ **Neither the Type-I or the Type-II FSF have exactly linear phase.**
- ▶ They have isolated zeros located at $z = r\cos(2\pi k/N)$, when $\phi_k = 0$,
 - with no accompanying reciprocal zeros located outside the unit circle at $z = 1/[r\cos(2\pi k/N)]$,
 - This causes phase nonlinearity.
 - Their group delays can have a peak-to-peak fluctuation of up to two sample periods ($2/f_s$).
- ▶ Type-I and -II FSFs are the most common types described in the literature of FSFs.
 - Their nonlinear phase has not received sufficient attention or analysis.
- ▶ Let's take steps to obtain linear phase by repositioning the isolated zero.

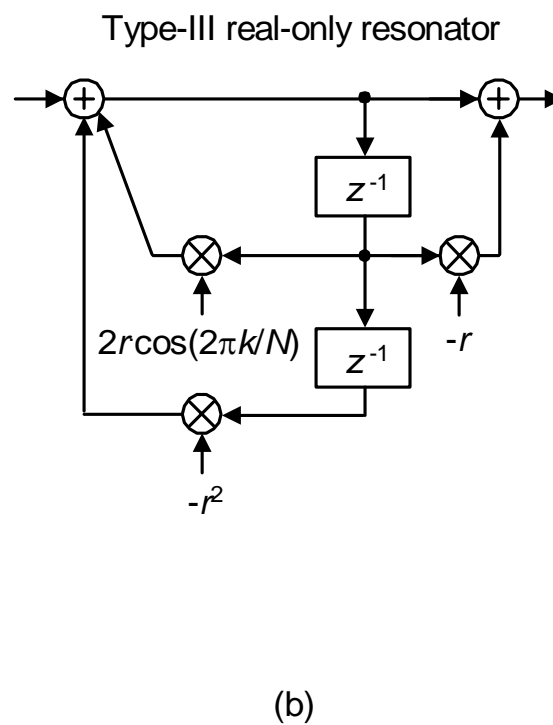
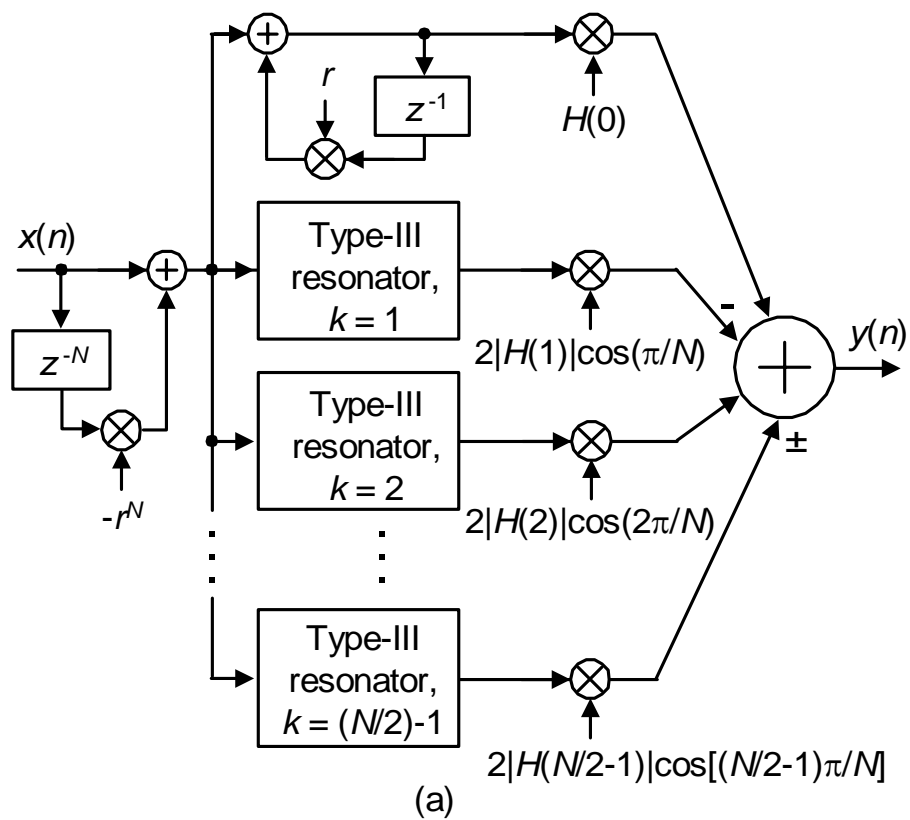
Linear-phase Multisection Real-Valued FSFs

- ▶ We can achieve exact real-FSF phase linearity by:
 - by setting $\phi_k = \pi k/N$.
 - This moves the Type-I FSF's isolated zero on top of the comb filter's zero located at $z = r$,

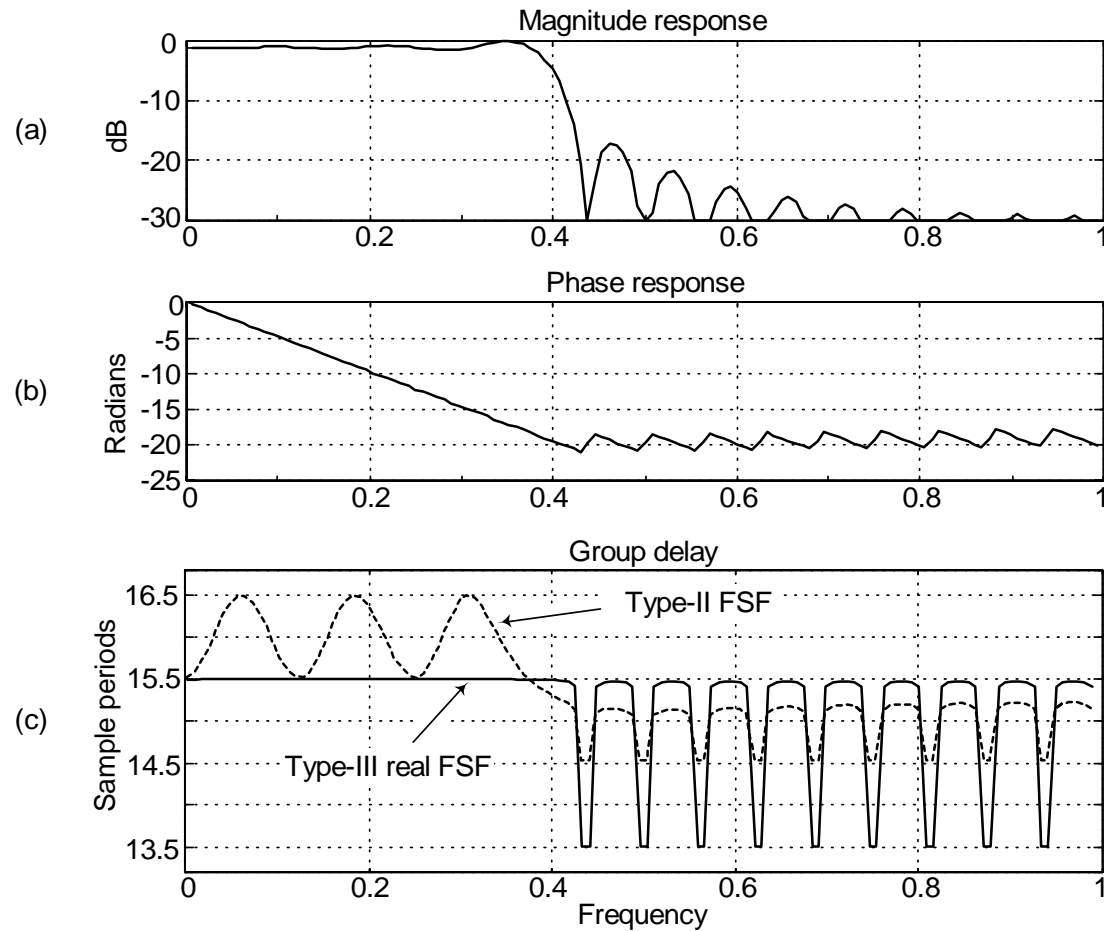
- ▶ The transfer function of this linear-phase *Type-III real FSF* is

$$H_{\text{Type-III}}(z) = (1 - r^N z^{-N}) \left[\frac{|H(0)|}{1 - rz^{-1}} + \sum_{k=1}^{N/2-1} \frac{2(-1)^k |H(k)| \cos(\pi k/N) [1 - rz^{-1}]}{1 - [2r \cos(2\pi k/N)]z^{-1} + r^2 z^{-2}} \right].$$

► The implementation of the linear-phase Type-III real FSF is

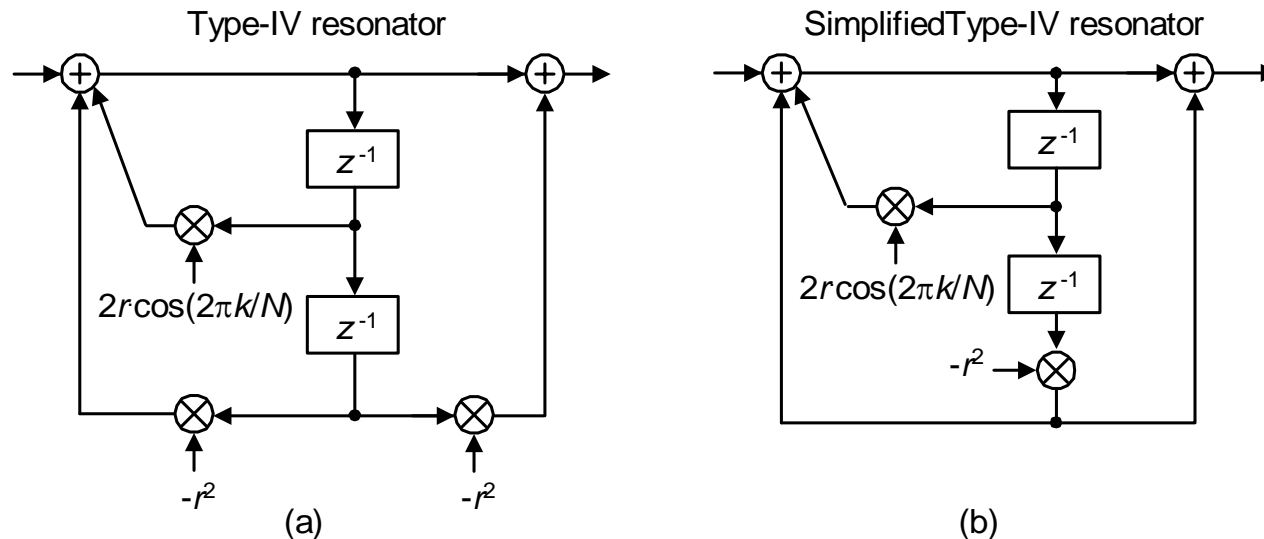


- ▶ Here's the frequency-domain performance of an eight-section Type-III FSF,
 - for $N = 32$ where the eight sections begin at DC ($0 \leq k \leq 7$).
 - Notice the improved Type-III phase linearity, a constant group delay of $(N-1)/2$ samples.



An Proposed Real-valued FSF

- ▶ There are many real-valued resonators that can be used in FSFs.
- ▶ Of particular interest is the proposed *Type-IV resonator*, shown as:



- ▶ This resonator deserves our attention because it:

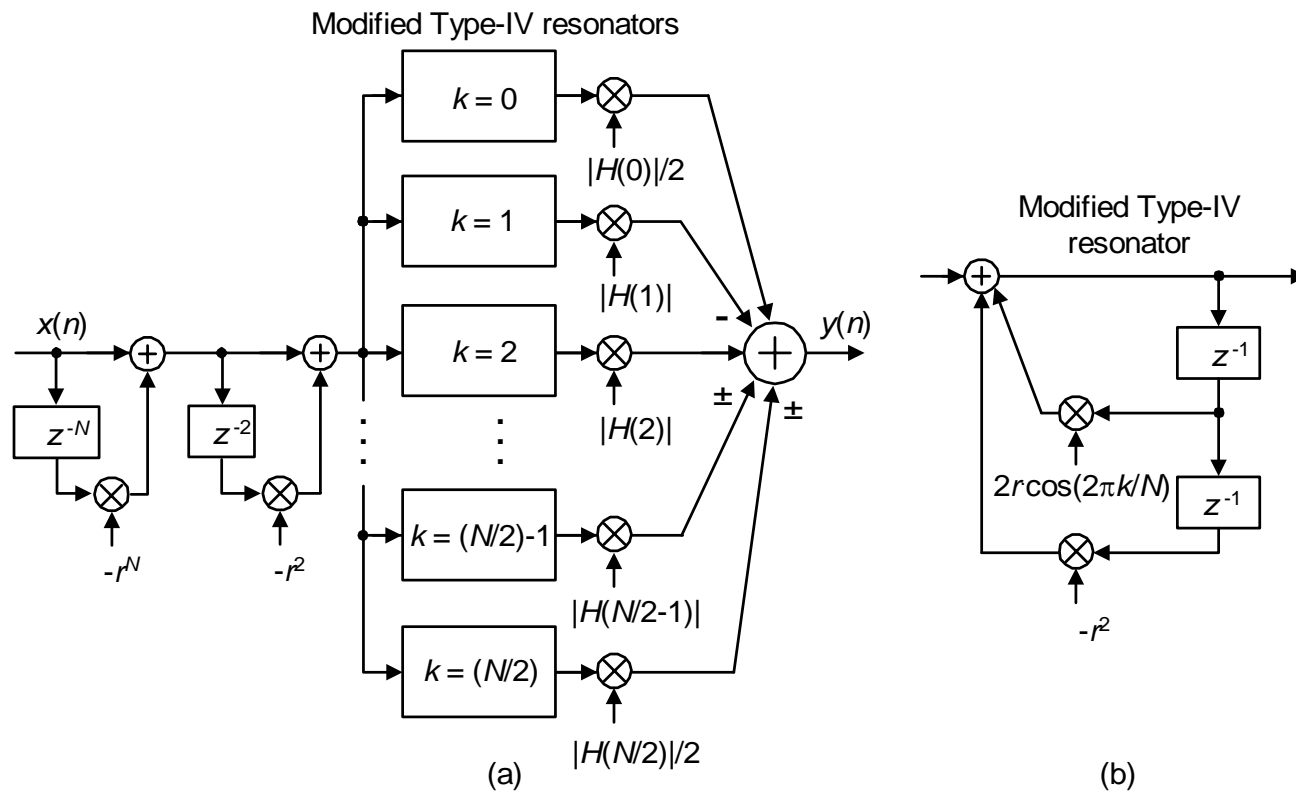
- is guaranteed-stable, exhibits highly linear phase,
- is computationally efficient, can implement highpass FIR filters, and
- yields better stopband attenuation performance than Type-I, -II, & -III FSFs.

► The **Type-IV FSF** transfer function is

$$H_{\text{Type-IV}}(z) = (1 - r^N z^{-N}) \sum_{k=0}^{N/2} \frac{(-1)^k |H(k)| (1 - r^2 z^{-2})}{1 - 2r \cos(2\pi k/N) z^{-1} + r^2 z^{-2}}$$

► We reduce the number of multiply and addition operations by

- by implementing the $(1 - r^2 z^{-2})$ term in the numerator as a second-order comb filter as shown below.



- ▶ The ' \pm ' symbols in remind us when N is even, $k = N/2$ could be odd or even.
- ▶ The Type-IV FSF frequency response is

$$H_{\text{Type-IV}}(e^{j\omega}) = e^{j\omega N/2} \sum_{k=0}^{N/2} \frac{(-1)^k |H(k)| [\cos(\omega N/2 - \omega) - \cos(\omega N/2 + \omega)]}{\cos(2\pi k/N) - \cos(\omega)}.$$

- ▶ The even- N peak resonance magnitude response for a single Type-IV FSF section is

$$|H_{\text{Type-IV}}(e^{j\omega})|_{\omega=2\pi k/N} = N, \quad k = 1, 2, \dots, \frac{N}{2} - 1$$

- and resonant magnitude gains at $k = 0$ (DC), and $k = N/2$ ($f_s/2$), are

$$|H_{\text{Type-IV}}(e^{j\omega})|_{\omega=0} = |H_{\text{Type-IV}}(e^{j\omega})|_{\omega=\pi} = 2N.$$

- ▶ We've covered a lot of ground so far.
- ▶ Here's a summary of real-valued FSFs.

Summary of even- N real FSF properties

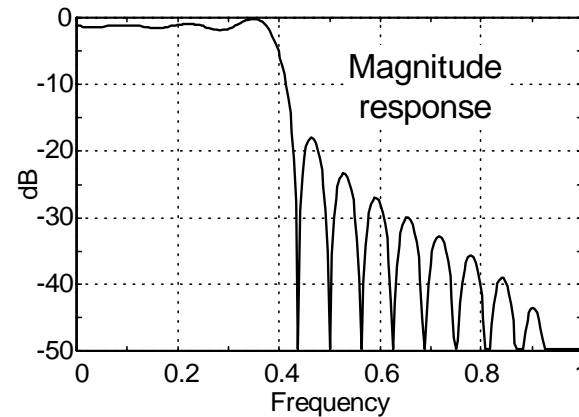
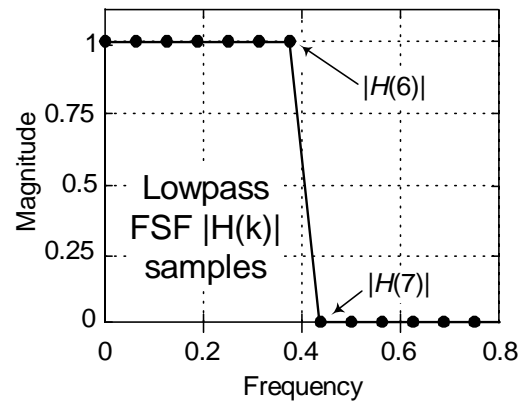
Real FSF type	Group delay	Multiplies	Adds	Remarks
Type-I	$\frac{N}{2}$	5	3	Real-coefficient FSF. Phase is only moderately linear.
Type-II	$\frac{N}{2}$	3	3	Modified, and more efficient, version of the Type-I FSF. Phase is only moderately linear. A binary left shift may eliminate one resonator multiply.
Type-III	$\frac{N-1}{2}$	4	3	Very linear phase. Cannot implement a linear-phase highpass filter.
Type-IV	$\frac{N}{2}$	2	2	Very linear phase. Improved stopband attenuation. Usable for lowpass, bandpass, or highpass filtering. Lean mean filtering machine.

- ▶ The section gain at their resonant frequencies is N for all four real-valued FSFs.

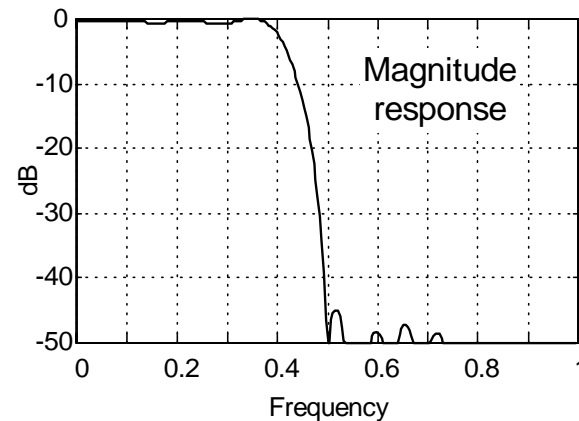
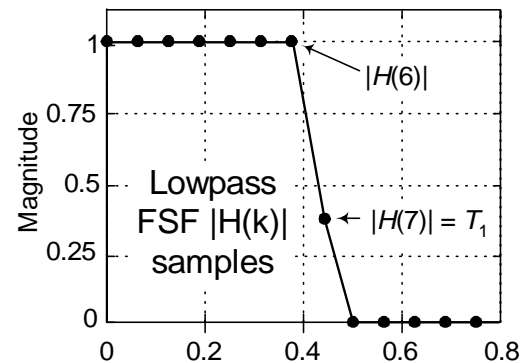
Improving Performance with Transition Band Coefficients

- ▶ We can increase FSF stopband attenuation defining non-unity $|H(k)|$ magnitude samples - in the transition region, between the passband and stopband.
- ▶ Consider a seven-section lowpass Type-IV FSF with $N = 32$.

No transition coefficient used.



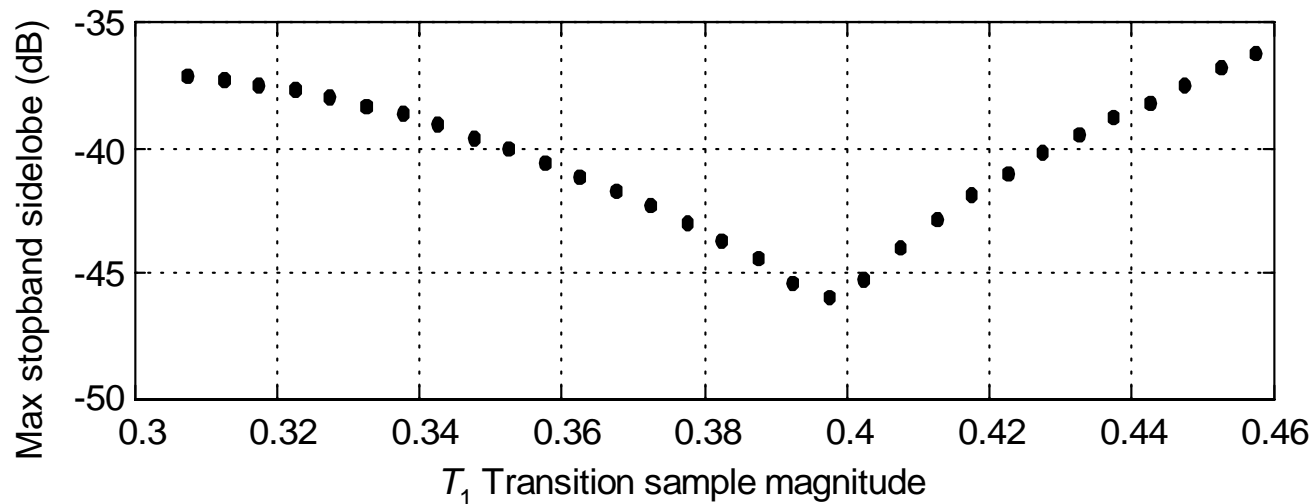
Single transition coefficient used.



- ▶ **Setting coefficient $T_1 = 0.389$ reduces the abruptness in the transition region.**
 - **Results in reduced passband ripple and the improved stopband attenuation.**
 - **Has computational cost of an additional FSF section and increased width of the transition region.**

- ▶ **Assigning a coefficient value of $T_1 = 0.389$ was not arbitrary nor magic.**

- ▶ **For the seven-section Type-IV FSF an optimum value for T_1 exists, as shown below.**



- ▶ **Further stopband attenuation is possible if two transition coefficients, T_1 and T_2 , are used,**
 - **such that $0 \leq T_2 \leq T_1 \leq 1$.**
 - **Each additional transition coefficient improves the stopband attenuation by roughly 25 dB.**

- ▶ **Finding the optimum values for the T coefficients is a daunting task.**

- ▶ **Optimum transition region coefficient values depend on:**
 - **the number of unity-gain FSF sections, the value of N , and**
 - **the number of transition coefficients used.**

- ▶ **There's no closed form equation available to calculate optimum transition coefficient values. ☹**

- ▶ **We must search for them empirically, but**
 - **commercial mathematical software packages have built-in *optimization* functions to facilitate this computationally-intensive search.**

▶ Happily, for the Type-IV FSF, tables of optimum transition coefficients are available.

- Compiled by the author and are provided in Reference [1]. 😊

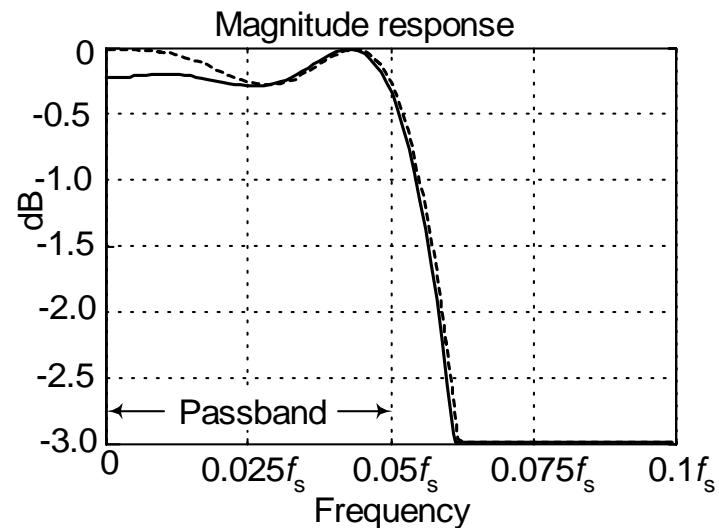
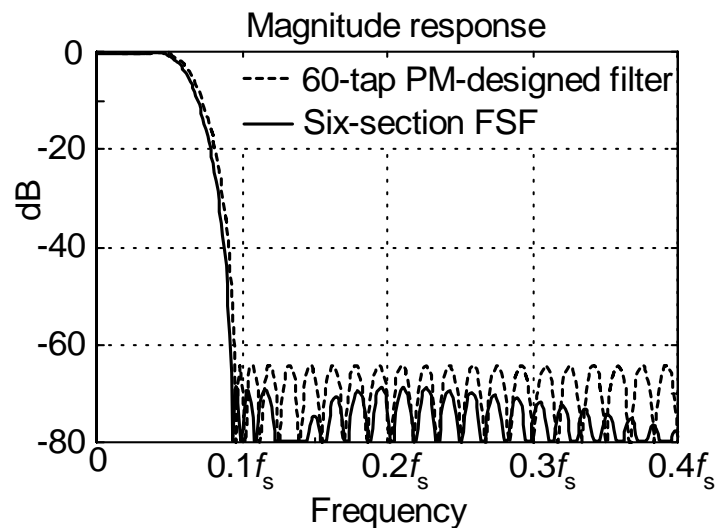
▶ The FSF design tables in Ref [1] look something like:

BW Atten T1 <i>N</i> = 16			BW Atten T1 <i>N</i> = 16			T2	
1	-44.9	0.41924081	1	-76.5	0.56626687	0.07922718	
2	-45.8	0.38969818	2	-77.2	0.55487263	0.08012238	
3	-47.3	0.36942214	3	-81.2	0.53095099	0.07087993	
4	-49.6	0.34918551	4	-87.7	0.49927622	0.05813368	

<i>N</i> = 24			<i>N</i> = 24				
1	-44	0.42452816	1	-73.6	0.57734042	0.08641861	
2	-44.1	0.40042889	2	-72.5	0.57708274	0.09305238	
3	-44.9	0.38622106	3	-72.9	0.56983709	0.09177956	
4	-45.7	0.37556064	4	-73.8	0.55958351	0.08770698	
5	-46.5	0.36663149	5	-75.6	0.54689579	0.08202772	

Type-IV FSF Example

- ▶ Consider the design of a linear-phase lowpass FIR filter with:
 - cutoff frequency of $0.05f_s$, stopband begins at $0.095f_s$,
 - max passband ripple of 0.3 dB (p-p), minimum stopband attenuation of 65 dB.
- ▶ A six-section Type-IV FSF with $N = 62$ and $r = 0.99999$, its frequency-domain performance satisfies our requirements and is the solid curve below.



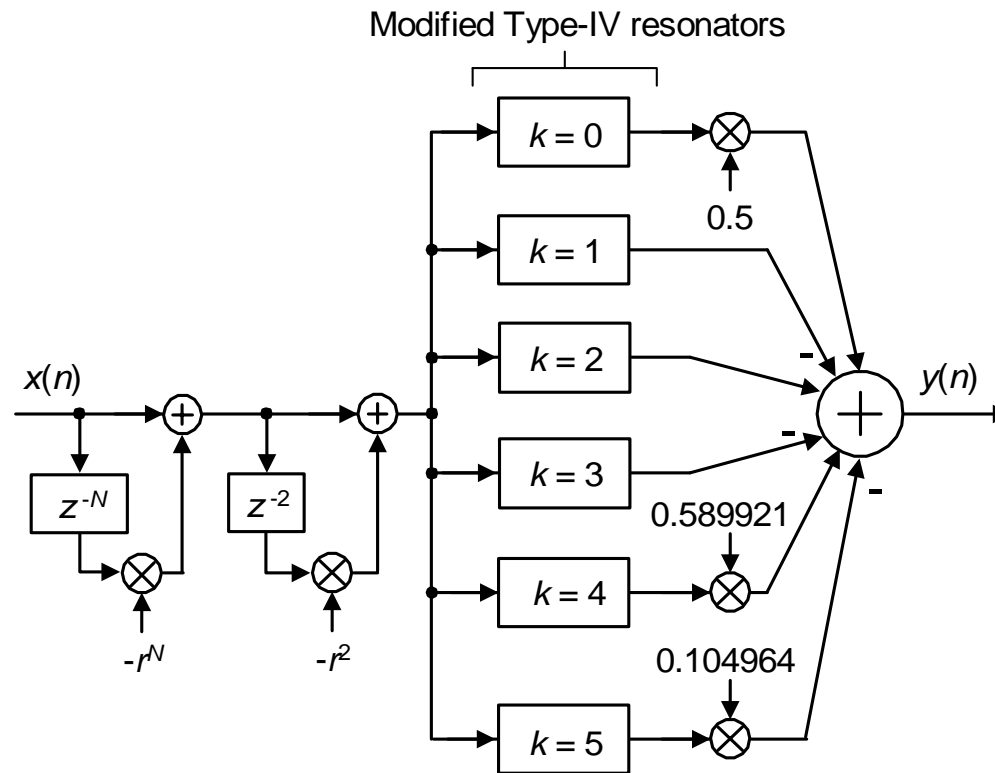
Two FSF transition region coefficients are:

- $|H(4)| = T_1 = 0.589921$, and $|H(5)| = T_2 = 0.104964$.

► A PM-designed filter implemented using a *folded* nonrecursive FIR structure

- requires 30 multiplies and 59 adds per output sample.

► The Type-IV FSF requires only 17 multiplies and 19 adds per output sample. 😊



When to Use an FSF

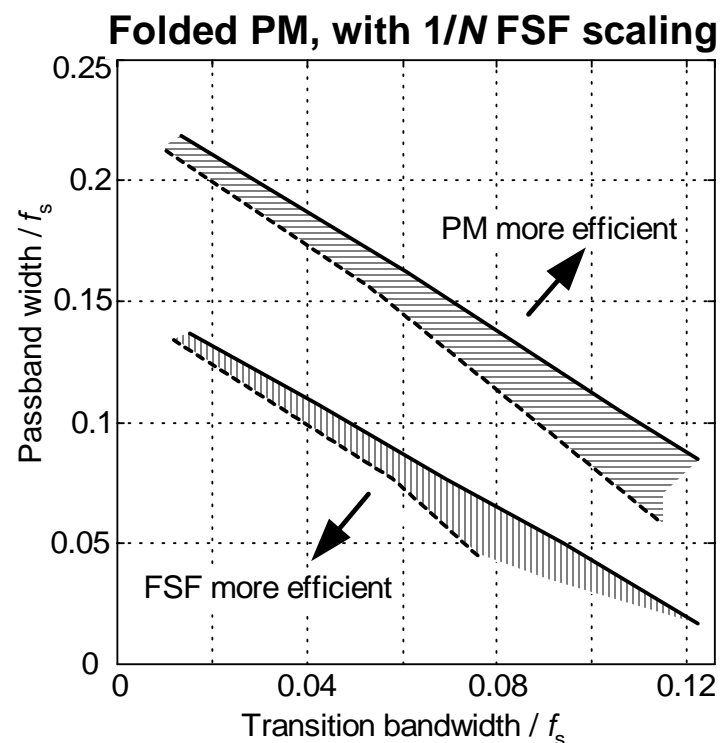
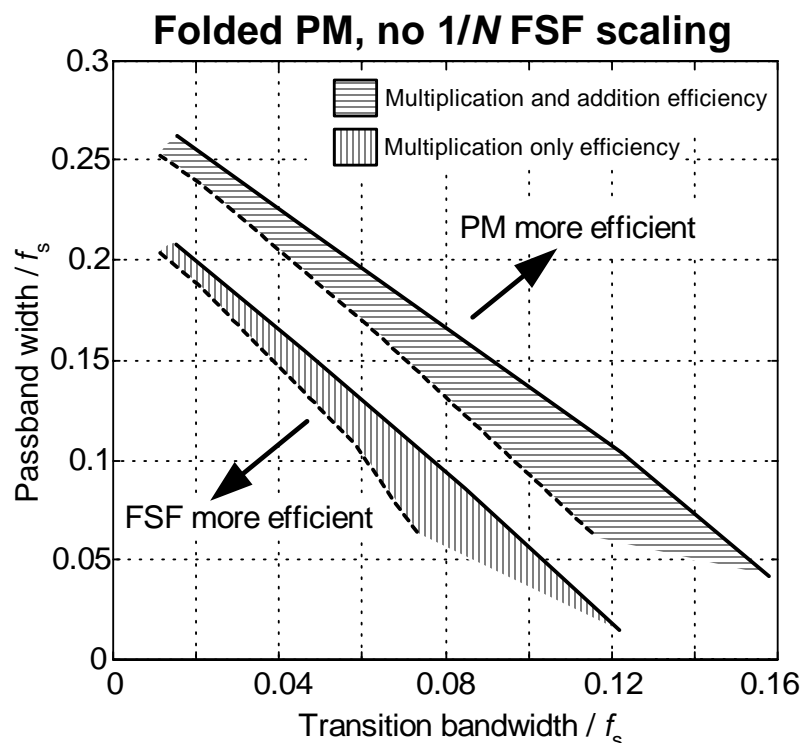
- ▶ In this section we answer the burning question;
 - when should you use a Type-IV FSF instead of a PM-designed FIR filter?

- ▶ The answer depends on:
 - the PM FIR filter implementation (folded, or non-folded),
 - do multiplications and additions require an equal number of clock cycles, and
 - whether FSF output scaling, by $1/N$, is needed.

- ▶ We have four scenarios to consider:
 - 1) Folded-structure FIR & no $1/N$ FSF scaling,
 - 2) Folded-structure FIR & $1/N$ FSF scaling used,
 - 3) Non-folded FIR & no $1/N$ FSF scaling, and
 - 4) Non-folded FIR & $1/N$ FSF scaling used.



► **Computational comparison between even- N Type-IV FSFs and *folded* PM-designed filters.**

- **The bands represent desired filter performance parameters where a Type-IV FSF and a *folded* PM designed filter have roughly equal computational workloads.**



► **If the desired FIR filter transition region width and passband width combination (a point) lies beneath the *comparison band of interest*, then a Type-IV FSF is more computationally efficient than a PM-designed filter.**

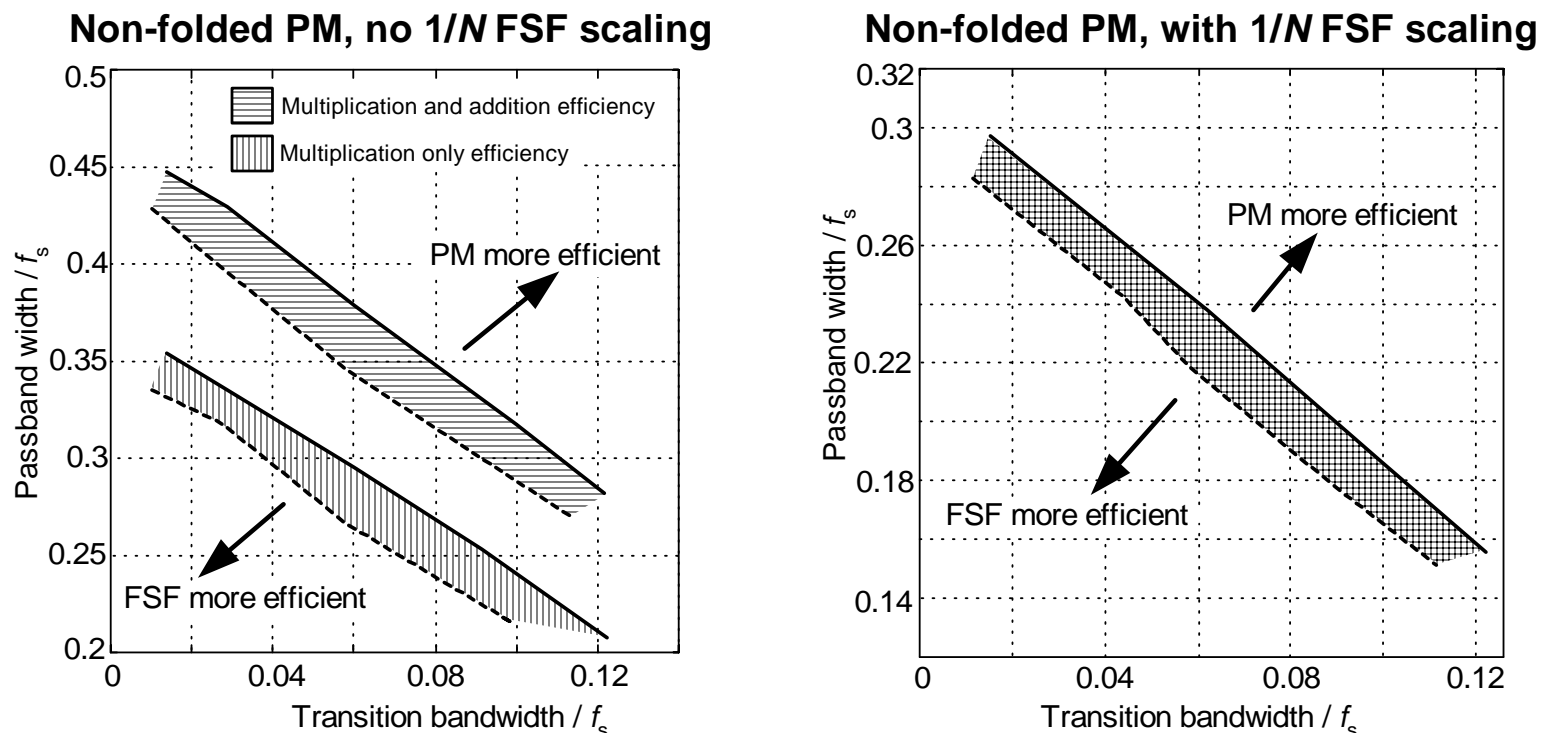
- **The solid lines are for three transition coefficients, and the dotted lines are for one transition coefficient.**

- ▶ For filter implementations where multiplies and adds require equal processor clock cycles,
 - the  band is the *comparison band of interest*.
- ▶ For filter implementations where a single multiply requires many more processor clock cycles than a single addition,
 - the  band is the *comparison band of interest*.
- ▶ The comparison charts assume an FSF damping factor of $r = 0.99999$.
- ▶ Performance criterion levied on the PM filter are typical Type-IV FSF properties (when floating-point coefficients are used) given in the following table.

Parameter	1-coefficient	2-coefficient	3-coefficient
Passband peak-peak ripple (dB)	0.7	0.35	0.16
Minimum stopband attenuation (dB)	-45	-68	-95

- ▶ FSFs have a gain of N ; in fixed point implementations the gain may cause overflow errors.
 - A gain reduction by a scaling factor of $1/N$ may be necessary (at resonators' outputs).
 - In this case, refer to the bands in the right panel of the previous page.

► **Computational comparison between even- N Type-IV FSFs and non-folded PM-designed filters.**



► **For a non-folded PM FIR filter, the number of multiplications & additions are roughly equal.**

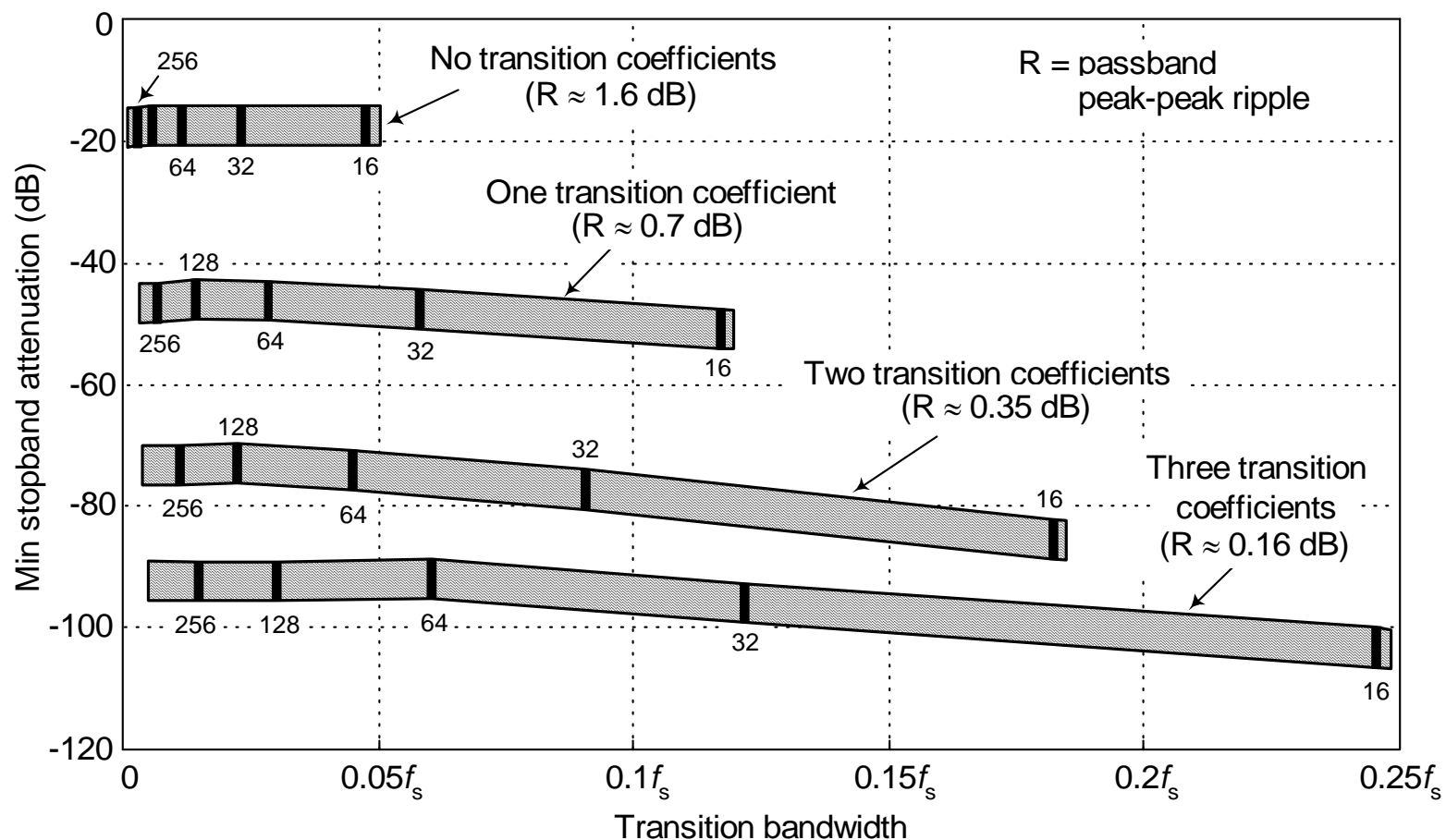
► **When $1/N$ scaling is used for the FSF, the number of multiplications & additions are equal.**

► **These conditions explain why the right panel in the above figure has only one band;**

- **for both filters, the number of *multiplications and additions* is roughly twice the number of *multiplication-only*.**

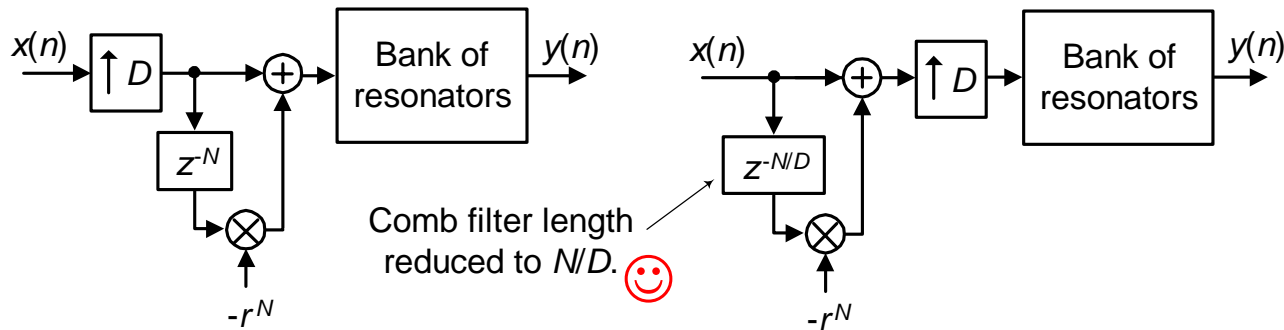
► To conclude our *When To Use an FSF Filter* discussion: an FSF should be considered if:

- your desired filter's passband & transition widths are below the implementation *comparison bands of interest* in the above figures, and
- your desired filter's stopband attenuation is within, or below, one of the bands in the following figure.

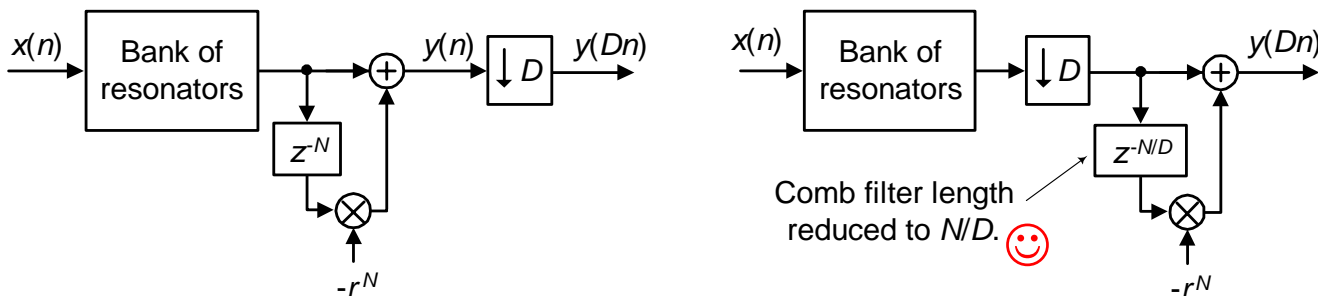


FSFs and With Sample Rate Conversion (SRC)

- ▶ FSFs need N memory locations to implement a length- N comb filter.
- ▶ For interpolation, comb filter storage can be reduced,
 - swap order of the "upsample by D " and the comb filter.



- ▶ Likewise for decimation, comb filter storage can be reduced,
 - swap order of the "downsample by D " and the comb filter.



- ▶ Of course D is chosen such that the ratio N/D is an integer.

FSF Summary

- ▶ **We've introduced the structure and performance of FSFs.**

- ▶ **Emphasis given to the practical issues**
 - **phase linearity,**
 - **stability, and**
 - **computational workload.**

- ▶ **FSFs are more efficient, for certain applications, than PM-designed FIR filters.**
- ▶ **FSFs are modular; their sections are computationally identical and well understood.**
- ▶ **Although recursive, FSFs can be designed to be stable and linear phase.**
- ▶ **Design tables of optimum transition region coefficients are available.**

- ▶ **More FSF details,**
 - **math derivations,**
 - **transition region coefficients design tables,**
 - **design examples,**
 - **design guidelines, and**
 - **additional literature references are provided in:**

Reference [1]:

Understanding Digital Signal Processing,
2nd Ed., by R. Lyons, Prentice Hall, Upper
Saddle River, New Jersey, 2004

