Fast Convolution Filtering -- From Basics to Filter Banks

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Overview

- Fast Convolution is
  - Multiplication in the frequency domain
  - A faster FIR filter for more than 10-30 taps
- Additional channels are even cheaper
- Other operations can be done in frequency domain
  - Downsampling/Decimation
  - Mixing*
- Put it all together and shake it all about. That's what Filter Banks are all about.

Goals

- Explain and demonstrate fast convolution filtering (FCF)
- Efficiently extend FCF to filter banks
  - Parallel filters
  - Mixing
  - Decimation

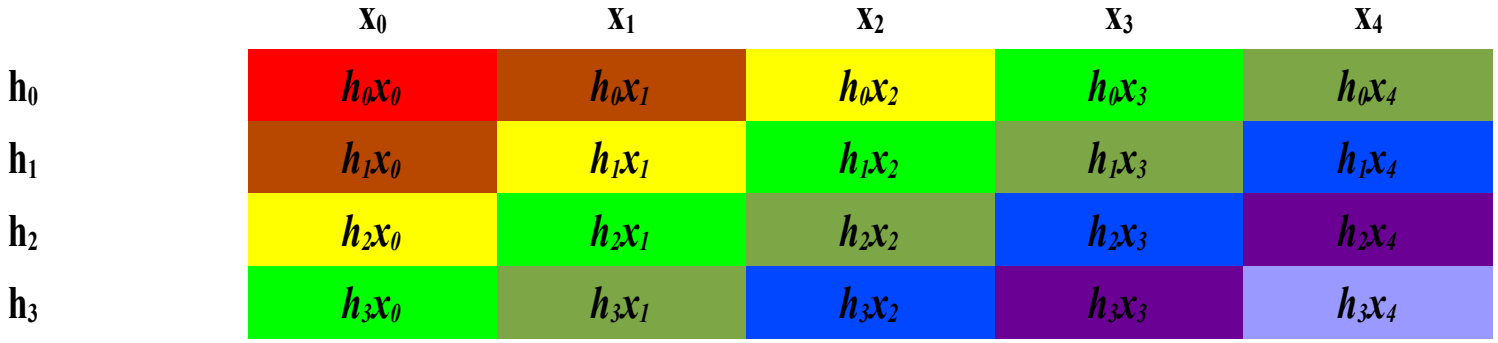
FIR Filtering is Convolution

$$y_n = h * x = \sum_m h_{n-m} x_m$$

- FIR filtering arguably the most important concept in DSP
  - Stable
  - Linear phase
- FIR filtering is another name for convolution
- Direct FIR filter costs don't scale well
  - Need twice the work for half the bandwidth
- Convolution of P elements with L elements is P+L-1 elements long

Visualizing Convolution

as the sum of outer product antidiagonals



$y_0 = h_0x_0$ Input-on Transient

$y_1 = h_1x_0 + h_0x_1$ Input-on Transient

$y_2 = h_2x_0 + h_1x_1 + h_0x_2$ Input-on Transient

$y_3 = h_3x_0 + h_2x_1 + h_1x_2 + h_0x_3$ Steady State

$y_4 = h_3x_1 + h_2x_2 + h_1x_3 + h_0x_4$ Steady State

$y_5 = h_3x_2 + h_2x_3 + h_1x_4$ Input-off Transient

$y_6 = h_3x_3 + h_2x_4$ Input-off Transient

$y_7 = h_3x_4$ Input-off Transient

Circular Convolution

- Accomplished by multiplying DFTs
- Periodic, just like DFT

	x_0	x_1	x_2	x_3	x_4
h_0	h_0x_0	h_0x_1	h_0x_2	h_0x_3	h_0x_4
h_1	h_1x_0	h_1x_1	h_1x_2	h_1x_3	h_1x_4
h_2	h_2x_0	h_2x_1	h_2x_2	h_2x_3	h_2x_4
h_3	h_3x_0	h_3x_1	h_3x_2	h_3x_3	h_3x_4

y_0	=	h_3x_2	+ h_2x_3	+ h_1x_4	+ h_0x_0		Wraparound Transient
y_1	=	h_3x_3	+ h_2x_4	+ h_1x_0	+ h_0x_1		Wraparound Transient
y_2	=	h_3x_4	+ h_2x_0	+ h_1x_1	+ h_0x_2		Wraparound Transient
y_3	=	h_3x_0	+ h_2x_1	+ h_1x_2	+ h_0x_3		Steady State
y_4	=	h_3x_1	+ h_2x_2	+ h_1x_3	+ h_0x_4		Steady State

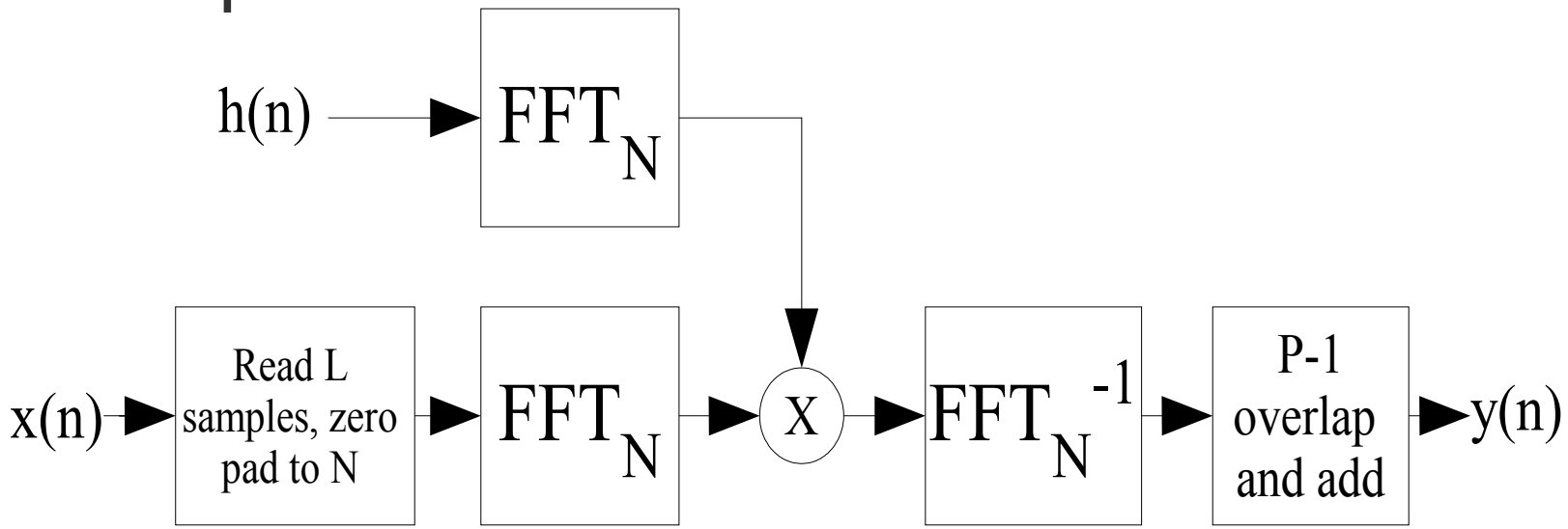
Making Circular Convolution Equivalent to Linear Convolution

- Zero-padding to P+L-1 avoids wraparound transient

	x_0	x_1	x_2	x_3	x_4	0	0	0
h_0	h_0x_0	h_0x_1	h_0x_2	h_0x_3	h_0x_4	0	0	0
h_1	h_1x_0	h_1x_1	h_1x_2	h_1x_3	h_1x_4	0	0	0
h_2	h_2x_0	h_2x_1	h_2x_2	h_2x_3	h_2x_4	0	0	0
h_3	h_3x_0	h_3x_1	h_3x_2	h_3x_3	h_3x_4	0	0	0

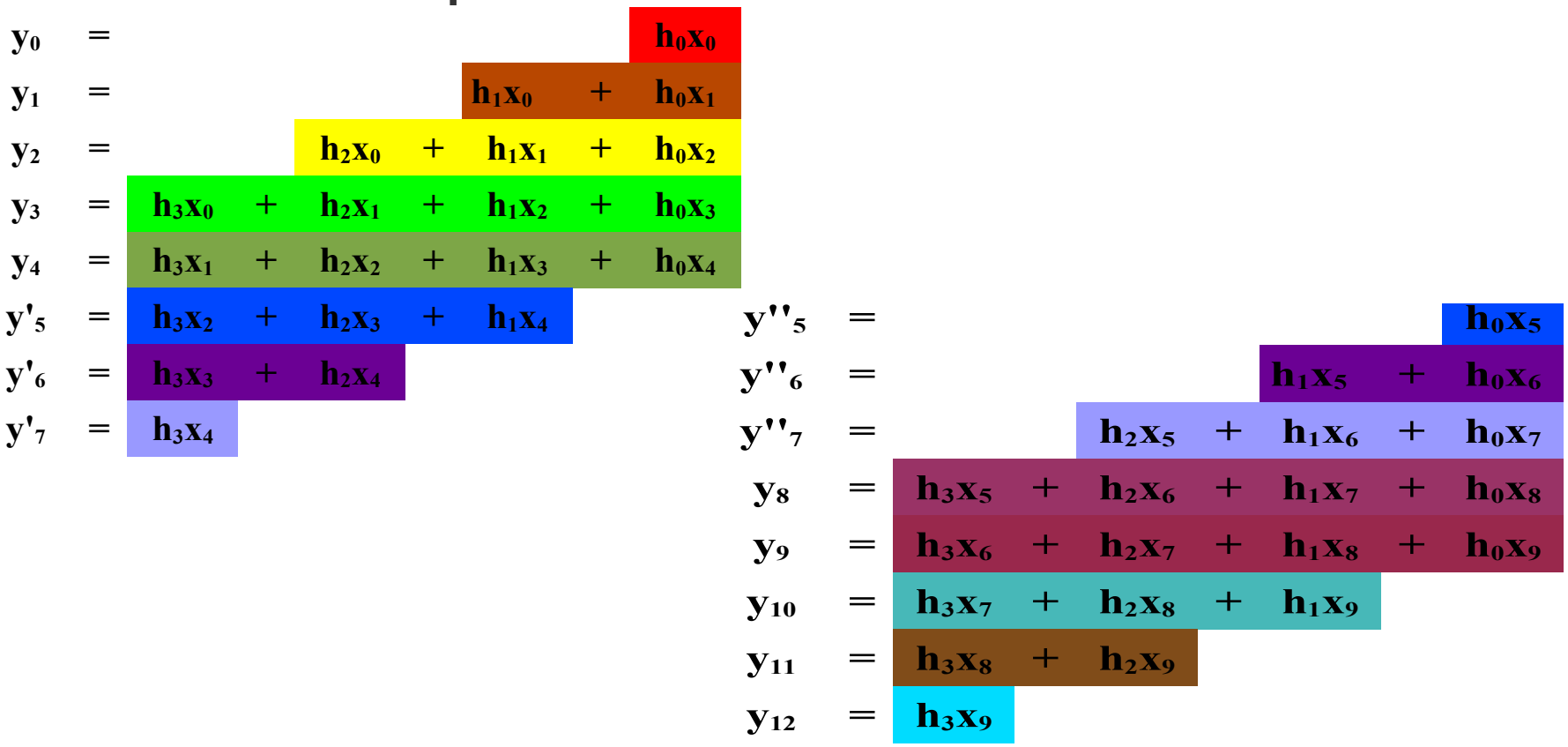
Overlap-Add Filtering

- Builds an arbitrarily long convolution from shorter convolutions, buffer-by-buffer
- Input-on transients from one buffer overlap (and add) with the input-off transients from the previous



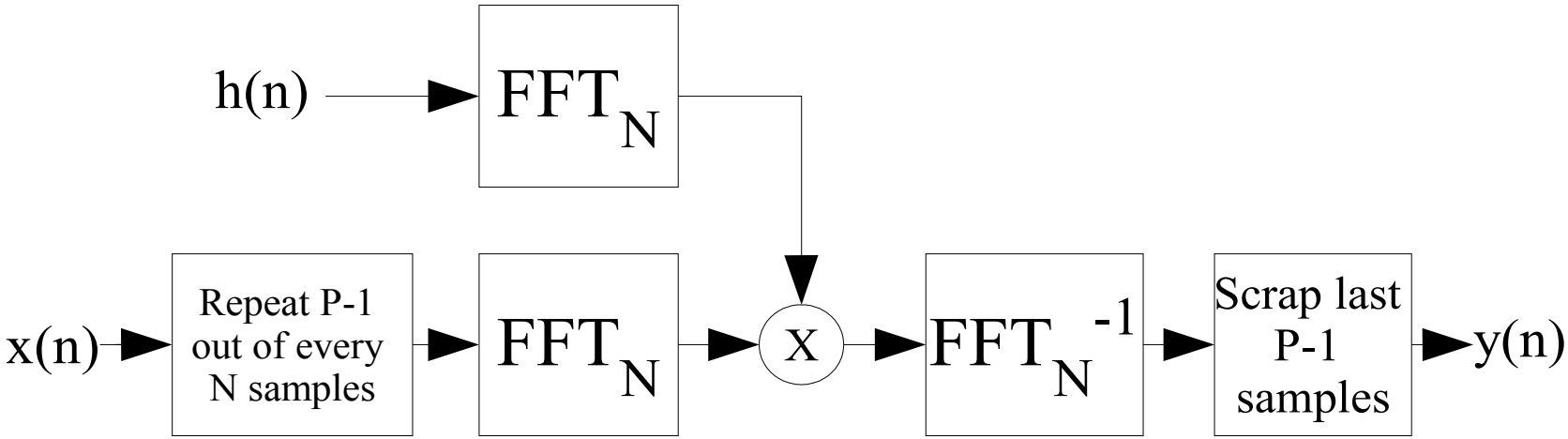
Overlap-Add (OA) Filtering

- Input-off transients complement the next buffer's input-on transients



Overlap-Save Filtering

- Eliminates addition of overlapping transients
- Circular convolution w/o zero padding
- Overlaps input buffers



Overlap-Save (OS) Filtering

- A.K.A. “Overlap-Scrap”
- Input buffers overlap
- wraparound transient is useless and discarded (i.e. scrapped)

$$
\begin{aligned}
y'_{s0} &= h_3x_2 + h_2x_3 + h_1x_4 + h_0x_0 && \text{Wraparound Transient}\\
y'_{s1} &= h_3x_3 + h_2x_4 + h_1x_0 + h_0x_1 && \text{Wraparound Transient}\\
y'_{s2} &= h_3x_4 + h_2x_0 + h_1x_1 + h_0x_2 && \text{Wraparound Transient}\\
y_3 &= h_3x_0 + h_2x_1 + h_1x_2 + h_0x_3 && \text{Steady State}\\
y_4 &= h_3x_1 + h_2x_2 + h_1x_3 + h_0x_4 && \text{Steady State}
\end{aligned}
$$

$$
\begin{aligned}
y'_{s2} &= h_3x_4 + h_2x_5 + h_1x_6 + h_0x_2 && \text{Wraparound Transient}\\
y'_{s3} &= h_3x_5 + h_2x_6 + h_1x_2 + h_0x_3 && \text{Wraparound Transient}\\
y'_{s4} &= h_3x_6 + h_2x_2 + h_1x_3 + h_0x_4 && \text{Wraparound Transient}\\
y_5 &= h_3x_2 + h_2x_3 + h_1x_4 + h_0x_5 && \text{Steady State}\\
y_6 &= h_3x_3 + h_2x_4 + h_1x_5 + h_0x_6 && \text{Steady State}
\end{aligned}
$$

OA & OS Comparison

	Overlap-Add Filtering	*Overlap-Save Filtering*
FFT input buffer	L input samples, zero-padded to N=L+P-1	N input samples, P-1 of which are repeated from the previous buffer
IFFT output buffer	The last P-1 samples from the previous output are added to the first P-1 samples of the current output. The first L (L = N-P+1) samples are then suitable for output.	The first P-1 samples of each IFFT output are discarded. The remaining N-P+1 samples are used as output.
Speed	Cost is logarithmic wrt. FIR length. Faster than direct for filters longer than 25-30 taps	Same scalability as OA, but slightly more efficient.
Name notes	The input-on and input-off transients are **overlap**ped and **add**ed.	The input buffers are **overlap**ping, i.e. some of the input is **save**d between iterations. [1]

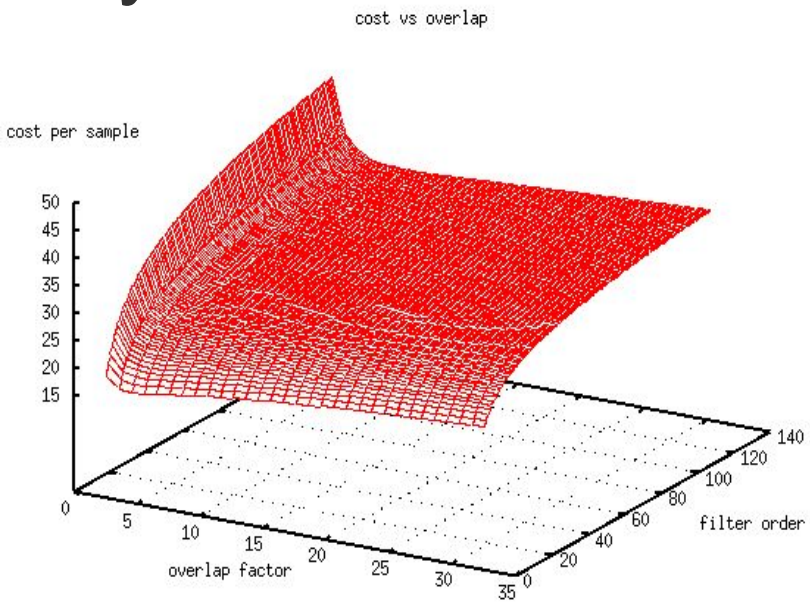
1 Alternately, the wraparound transient is “scrapped”.

Fast Convolution Complexity

- filter order P'=P-1
- Define overlap factor V=N/P'
- Cost per FFT buffer = O(VP' *log*(VP'))
- Throughput samples per buffer =(V-1)P'
- Cost per sample = O(V *log*(VP') / (V-1))
- For a given overlap factor V
  - V determines multiplier against big “O”
  - Cost per sample = O(log(P'))

Determining Overlap Factor V

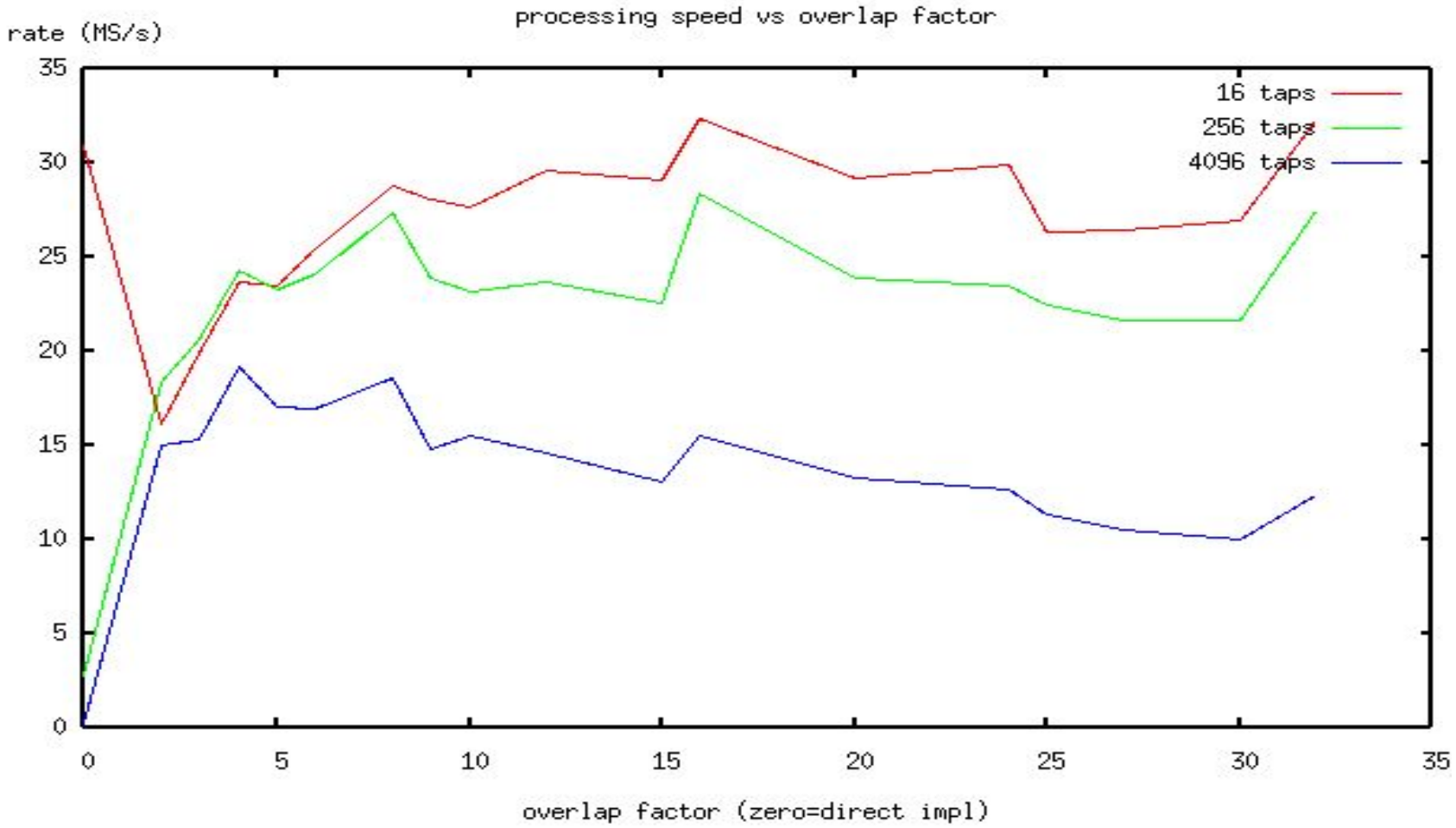
- Cost per sample is O(V *log*(VP') / (V-1))
- For a given filter order P', how is the fastest V determined?
  - Theoretically: find minimum of V *log*(VP') / (V-1)



  - Empirically: Benchmark!

Overlap Factor Efficiency

- Theory vs. practice
- If possible, compare FCF and direct FIR filtering (shown at 0 in figure)



Recap

- Multiplication in the frequency domain is equivalent to convolution in time domain
- Circular Convolution is faster (i.e. more scalable) than linear convolution
- Fast Convolution combines smaller convolutions to calculate a longer convolution